# APPROXIMATE LIE SYMMETRIES

BY

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*To my beloved parents,brothers and sisters*

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All thanks and praises to almighty Allah, Who had helped me to complete this work. And peace be upon his last and greatest prophet, Muhammed, sallallaaahu alayhi wa sallam, who has been sent for guidance, a missionary and a harbinger of truth to all mankind.

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# THESIS ABSTRACT

**NAME:** Waheed Abdelwahab Ahmed

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*In this thesis, we consider partial differential equations with a small parameter. By employing different approximate Lie symmetry approaches we attempt to find solutions to such equations. We compare these different approaches and discuss advantages of using one over the other.*

# ملخص الرسالة

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هذه الرسالة البحثية أهتمت بدراسة المعادلات التفاضلية التى تحتوى على معّلمة صغيرة ، و ذلك من خلال توظيف طرق مختلفة لتقريب تماثلات **لى** و من ثم الحصول على حلول لتلك المعادلات و أيضا مقارنة هذه الطرق ومناقشة أفضلية إستخدام طريقة على الاخرى.

# CHAPTER 1

# INTRODUCTION

## 1.1 Inroduction

The nonlinear equations occur frequently in many physical, biological, engineering and financial problems. To find analytical solutions to such problems is a challenging matter. As there are many numerical approaches to obtain approximate solutions the importance of such analytical solutions is due to the fact that they provide a benchmark for testing numerical schemes which could be used to obtain numerical solutions. Perturbation technique has been widely used to tackle such problems. This relies on the presence of a small parameter or a term which could be ignored to obtain an unperturbed equation, solution of which is known. The solution to the original unperturbed problem can then be found by an iterative procedure. Another method which is suitable for many nonlinear differential equations is to use the Lie symmetry analysis to either reduce the number of independent variables or to reduce the order. The Lie symmetry approach provides a systematic and unified way to obtain exact solutions through one of the above means. Although the idea itself was given by Sophus Lie at the end of the nineteenth century, the use of this became more popular more recently. In the past two decades a number of authors have published numerous papers on this approach. One may refer to [5], [8] and [16] for a good account of the Lie symmetry approach and its applications. The focus of this study is to combine the perturbation and the Lie symmetry approach for nonlinear partial differential equations with a small parameter.

Approximate Lie symmetries is based upon the utilization of perturbation approach in finding the approximate symmetries of certain equations. Baikov,Gazizov and Ibragimov [4] proved approximate Lie theorem enable one to construct approximate symmetries that are stable under small perturbation of the differential equations. Fushchich, and Shtelen [12] and then Gazizov [13] considered such equations with a small parameter, introduced approximate symmetries and shown that the approximate symmetries form an approximate Lie algebra. Since then many authors have used the approximate Lie symmetries to study nonlinear pdes with a small parameter, see for example [6],[19],[17],[15] and the references therein. This concept of approximate symmetries and approximate invariants was studied for a wave equation with quadratic non-linearity [20], and a non-linear wave equation with a small parameter [12]. In an interesting paper [21] Pakdermirli and Yurusoy provided a comparison between different methods of using approximate symmetries. Baikov and Kordyukova [3] showed that all exact symmetries of the linear wave equation are inherited by the Boussinesq type equation with a small parameter. Valenti [25] calculated the approximate solution for a model describing dissipative media making use of the approximate generator of the first-order approximate symmetries. Bokhari, Kara and Zaman [6] considered some nonlinear evolution equations with a small parameter and their approximate symmetries. Grebener and Oberlack [14] considered the approximate Lie symmetries of the Navier-stokes equation while Qian and Wei [22] discussed the perturbed Burger equation from this point of view. Nadjafikhan and Mokhtary [19] discussed the Gardner equation with a small parameter. Diatta, Soh and Khalique[17] considered the approximate symmetries and solutions of the hyperbolic heat equation with variable parameters. Zhi-Yong, Yu-Fu and Xue-Lin [26] performed classification and approximate solutions to a class of perturbed nonlinear wave equation using the method originated from Fushchich and Shtelenand, In another paper [28] they introduced a new method to obtain approximate symmetry of nonlinear evolution equation from perturbations. Ruggieri and Valenti [23] studied the approximate symmetries of a mathematical model describing one dimensional media in a medium with a small nonlinear viscosity.

## 1.2 Lie Symmetries

A preliminary definition of Lie group and other notions that are used in this thesis are presented in what follows.

### 1.2.1 The Symmetry of Functions

**Definition 1.1** *A mathematical relation between variables is said to possess a symmetry property if one can subject the variables to a group of transformation and resulting expression reads the same in the new variables as the original expression. The relation is said to be invariant under the transformation.*

**Example 1** *Consider the transformation along horizontal lines*

$$T^{trans} = \begin{cases} x = \tilde{x} + s, \\ y = \tilde{y} \end{cases} \tag{1.1}$$

*(see Figure 1.1). By varying the transformation parameter $s$, we can move continuously and invertibly to any point $(x, y)$ on horizontal line.*

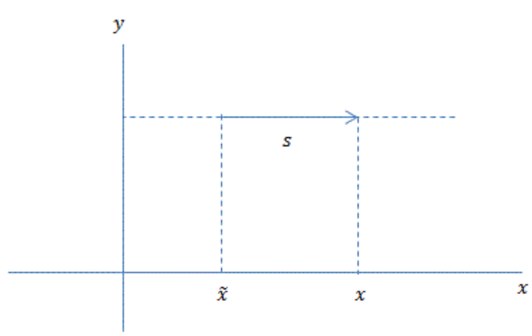


Figure 1.1: Mapping of points by a translation group.

### 1.2.2 One-Parameter Lie Group

**Definition 1.2** *Let the vector $x = (x^1, x^2, \ldots, x^n)$ lie in some continuous open set $D$ on the n-dimensional Euclidean manifold $\mathbb{R}^n$. Define the transformation*

$$T^{s} : \{\mathbf{x} = F[\tilde{\mathbf{x}}, s]\}. \tag{1.2}$$

*The functions $F^j$ are infinitely differentiable with respect to the real variables $\mathbf{x}$ and are analytic functions of the real continuous parameter $s$, which lies in an open interval $\mathbf{S}$.*

*The transformation $T^s$ is a one-parameter Lie group with respect to the binary operation of composition if and only if :*

*(i) There is an identity element $s \longrightarrow s_0$ such that $\tilde{\mathbf{x}}$ is mapped to itself:*

$$T^{s_0} : \{\tilde{\mathbf{x}} = F[\tilde{\mathbf{x}}, s_0]\}. \tag{1.3}$$

*(ii) For every value of $s$ there is an inverse $s \longrightarrow s_{inv}$ such that $\mathbf{x}$ is returned to $\tilde{\mathbf{x}}$:*

$$T^{s_{inv}} : \{\tilde{\mathbf{x}} = F[\mathbf{x}, s_{inv}]\}. \tag{1.4}$$

*(iii) The binary operation of composition produces a transformation that is a member of the group $T^{s_1}.T^{s_2} = T^{s_3}$ i.e., the group is closed. Consider two members of the group,*

$$T^{s_1} : \{\mathbf{x} = F[\tilde{\mathbf{x}}, s_1]\}. \tag{1.5}$$

*and*

$$T^{s_2} : \{\tilde{\mathbf{x}} = F[\tilde{\tilde{\mathbf{x}}}, s_2]\}. \tag{1.6}$$

*If we compose $T^{s_1}$ and $T^{s_2}$, the result is*

$$T^{s_3} : \{\mathbf{x} = F[\mathbf{F}[\tilde{\tilde{\mathbf{x}}}, s_1], s_2] = F[\tilde{\tilde{\mathbf{x}}}, s_3]\}. \tag{1.7}$$

*where $s_3 = \phi[s_1, s_2] \in S$. The function $\phi$ defining the law of composition of $T^s$ is an analytic function of $s_1 \in S$ and $s_2 \in S$ and is commutative ($s_3 = \phi[s_1, s_2] = \phi[s_2, s_1]$); thus Lie groups are Abelian.*

*(iv) The group is associative: $(T^{s_1}.T^{s_2}).T^{s_3} = T^{s_1}.(T^{s_2}.T^{s_3})$.*

### 1.2.3 Invariant Functions

Central to all of the development of symmetry theory is the concept of an invariant function.

**Definition 1.3** *A function $\psi[x]$ is said to be invariant under the Lie group $T^s : \{x^j = F^j[\tilde{\mathbf{x}}, s], j = 1, \dots, n\}$ if and only if*

$$\psi[\mathbf{x}] = \psi[F[\tilde{\mathbf{x}}, s]] = \psi[\tilde{\mathbf{x}}]. \tag{1.8}$$

*For invariance, the parameter $s$ must vanish from the transformation so that the function reads the same in the new variables.*

### 1.2.4 Infinitesimal Form of a Lie Group

In the last section we defined a one-parameter Lie group of the form

$$\tilde{x}^j = F^j[\mathbf{x}, s], \tag{1.9}$$

Now expand (1.9) in a Taylor series about $s = 0$:

$$\tilde{x}^j = x^j + s\left[\frac{\partial F^j}{\partial s}\right]_{s=0} + O(s^2) + \ldots, \qquad j = 1, \ldots, n. \tag{1.10}$$

The derivatives of the various $F^j$ with respect to the group parameter $s$ evaluated at $s = 0$ are called the $infinitesimals$ of the group and are traditionally denoted by

$$\xi^j[\mathbf{x}] = \left[\frac{\partial}{\partial s}F^j[\mathbf{x}, s]\right]_{s=0}, \qquad j = 1, \ldots, n. \tag{1.11}$$

### 1.2.5 Lie Series,The Group Operator, and the Infinitesimal Invariance Condition for Functions

The condition for invariance of the function given in section (1.2.3) is difficult to apply in practice because of the usually nonlinear dependence of $F^j$ on the group parameter. The condition given in section (1.2.3) requires that the transformation be substituted into $F^j$ and then rearranged to read like the original function, This can be an extremely weary procedure for testing the invariance condition of complicated functions such as differential equations. We need something equivalent to (1.8) but much simpler to apply. Substitute (1.9) into the analytic function

$\psi[\tilde{\mathbf{x}}]$ :

$$\psi[\tilde{\mathbf{x}}] = \psi[F[\mathbf{x}, s]]. \tag{1.12}$$

We now expand (1.12) in a Tyalor series about the identity element $s = 0$:

$$\psi[\tilde{\mathbf{x}}] = \psi[\mathbf{x}] + s\left[\frac{\partial\psi}{\partial s}\right]_{s=0} + \frac{s^2}{2!}\left[\frac{\partial^2\psi}{\partial s^2}\right]_{s=0} + \frac{s^3}{3!}\left[\frac{\partial^3\psi}{\partial s^3}\right]_{s=0} + \ldots. \tag{1.13}$$

Using the chain rule, we obtain

$$\left[\frac{\partial\psi}{\partial s}\right]_{s=0} = \sum_{i=1}^{n} \frac{\partial\psi}{\partial F^j}\left[\frac{\partial F^j}{\partial s}\right]_{s=0} = \sum_{i=1}^{n} \xi^j \frac{\partial\psi}{\partial f^j} \tag{1.14}$$

we shall use Einstein notation and write instead $\xi^j \frac{\partial\psi}{\partial f^j}$.

The equation (1.14) becomes the Lie series representation of the function $\psi$:

$$\psi[\tilde{\mathbf{x}}] = \psi[\mathbf{x}] + s\left(\xi^j \frac{\partial\psi}{\partial x^j}\right) + \frac{s^2}{2!}\xi^j \frac{\partial}{\partial\xi^j}\left(\xi^{j_1}\frac{\partial\psi}{\partial\xi^{j_1}}\right) + \frac{s^3}{3!}\xi^j\frac{\partial}{\partial\xi^j}\left(\xi^{j_1}\frac{\partial}{\partial\xi^{j_1}}\left(\xi^{j_2}\frac{\partial\psi}{\partial\xi^{j_2}}\right)\right) + \ldots \tag{1.15}$$

where $j_1, j_2 \ldots$ vary from 1 to 4.

The condition $\psi[\tilde{\mathbf{x}}] = \psi[\mathbf{x}]$ is satisfied if and only if $\xi^j \frac{\partial\psi}{\partial x^j} = 0$, that is $\sum_{i=1}^{n} \xi^j \frac{\partial\psi}{\partial x^j} = 0$.

**Theorem 1.1** *The analytic function $\psi[\mathbf{x}]$ is invariant under the Lie group $T^s : \{\tilde{x}^j = F^j[\mathbf{x}, s], j = 1, \ldots, n\}$ or, equivalently, the infinitesimal group $\xi^j[\mathbf{x}]$, j=1,...,n, if and only if $\psi[\mathbf{x}]$ satisfies the condition*

$$\xi^j[\mathbf{x}]\frac{\partial\psi}{\partial x^j} = 0 \tag{1.16}$$

*The operator*

$$X \equiv \xi^j[x]\frac{\partial}{\partial x^j} \tag{1.17}$$

*is called the group operator (or the group generator) and $X\psi$ is called the Lie derivative (or the determining equation) of $\psi$.*

### 1.2.6 Solving the Characteristic Equation ($X\psi[\mathbf{x}] = 0$)

The linear first-order PDE (1.16) has an associated system of $n-1$ characteristic first-order ODEs of the form

$$\frac{dx^1}{\xi^1[\mathbf{x}]} = \frac{dx^2}{\xi^2[\mathbf{x}]} = \frac{dx^3}{\xi^3[\mathbf{x}]} = \cdots = \frac{dx^n}{\xi^n[\mathbf{x}]} \tag{1.18}$$

with integrals

$$\psi^i = \psi^i[\mathbf{x}], \qquad i = 1, \ldots, n-1 \tag{1.19}$$

which are invariants of the group. Each of the function $\psi^i$ represents an infinite family of curves(or surfaces), one for each possiple value of $\psi^i$. Every curve $\psi = constant$ is individually invariant under the group.

**Example 2** *(The rotation group in two dimensions) Consider the rotation group*

$$T^{rot} = \begin{cases} \tilde{x} = x\cos[s] - y\sin[s] \\ \tilde{y} = x\sin[s] + y\cos[s] \end{cases} \tag{1.20}$$

*The infinitesimals of the group are* $(\xi, \eta) = (-y, x)$*, and the invariance condition is*

$$-y\frac{\partial \psi}{\partial x} + x\frac{\partial \psi}{\partial u} = 0 \tag{1.21}$$

*with corresponding characteristic equation*

$$\frac{dy}{x} = -\frac{dx}{y} \tag{1.22}$$

*Equation* (1.22) *in the two terms are uncoupled. This gives the integral invariant which are the family of circles*

$$\psi = \psi(x, y) = x^2 + y^2. \tag{1.23}$$

### 1.2.7 Point Transformations and Extended Transformations(Prolongation)

In order to apply transformations (1.2) to an nth order differential equation, the coresponding infinitesimal symmetry generator (1.17) needs to be extended or prolonged to nth order. Consider the finite one-parameter Lie group. $T^s$, in several variables:

$$T^s = \begin{cases} \tilde{x}^j = F^j[\mathbf{x}, \mathbf{y}, s], & j = 1, \dots, n \\ \tilde{y}^i = G^i[\mathbf{x}, \mathbf{y}, s], & i = 1, \dots, m \end{cases} \tag{1.24}$$

**Finite Transformation of First Partial derivatives**

The first partial derivatives is required to satisfy the first-order contact condition (invariant condition)

$$d\tilde{y}^i - \tilde{y}^i_\alpha d\tilde{x}^\alpha = 0 \qquad \alpha = 1, \ldots, n \tag{1.25}$$

that is $d\tilde{y}^i - \sum_{\alpha=1}^{n} \tilde{y}^i_\alpha d\tilde{x}^\alpha = 0 \qquad i = 1, \ldots, n.$

To prolong the group; we are taking differentials of (1.24) as

$$\begin{aligned} d\tilde{x}^\alpha &= (D_\beta F^\alpha) dx^\beta \\ d\tilde{y}^\alpha &= (D_\beta G^i) dx^\beta \end{aligned} \tag{1.26}$$

substituting (1.26) into (1.25):

$$(D_\beta G^i - \tilde{y}^i_\alpha D_\beta F^\alpha) dx^\beta = 0 \tag{1.27}$$

The differentials $dx^\beta$ are independent quantities. Therefore, in order for (1.27) to be satisfied, the expression in parentheses must be zero:

$$D_\beta G^i - \tilde{y}^i_\alpha D_\beta F^\alpha = 0, \qquad i = 1, \ldots, m \tag{1.28}$$

We assume that the Jacobian of the transformation in nonzero, $\|D_\beta F^\alpha\| \neq 0$. Then the inverse of $D_\beta F^\alpha$ exits such that

$$D_\beta F^\alpha (D_j F^\beta)^{-1} = \delta^\alpha{}_j. \tag{1.29}$$

Right-multiplying both sides in (1.28) by $(D_j F^\beta)^{-1}$:

$$D_\beta G^i (D_j F^\beta)^{-1} - \tilde{y}^i_\alpha D_\beta F^\alpha (D_j F^\beta)^{-1} = 0, \tag{1.30}$$

or

$$D_\beta G^i (D_j F^\beta)^{-1} - \tilde{y}^i_\alpha \delta_j{}^\alpha = 0. \tag{1.31}$$

Noting that $\tilde{y}^i_j = \tilde{y}^i_\alpha \delta^\alpha{}_j$, the finite transformation of first partial derivatives is determined:

$$\tilde{y}^i_j = D_\beta G^i (D_j F^\beta)^{-1}. \tag{1.32}$$

The once extended finite transformation is

$$\begin{aligned} \tilde{x}^j &= F^j[\mathbf{x}, \mathbf{y}, s], \quad j = 1, \ldots, n \\ \tilde{y}^i &= G^i[\mathbf{x}, \mathbf{y}, s], \quad i = 1, \ldots, m \\ \tilde{y}^i_j &= G^i{}_{\{j\}}[\mathbf{x}, \mathbf{y}, \mathbf{y}_1, s] \end{aligned} \tag{1.33}$$

Where $\mathbf{y}_1$ is the vector of all possible first partial derivatives and where

$$G^i{}_{\{j\}}[\mathbf{x}, \mathbf{y}, \mathbf{y_1}, s] = D_\beta G^i (D_j F^\beta)^{-1}. \tag{1.34}$$

**Finite transformation of second and higher partial derivatives:**

The once extended group (1.33) satisfies the second contact condition

$$d\tilde{y}^i_{j1} - \tilde{y}^i_{j\alpha} d\tilde{x}^\alpha = 0 \tag{1.35}$$

Taking differentials of (1.33):

$$\begin{aligned} d\tilde{x}^\alpha &= (D_\beta F^\alpha) dx^\beta, \\ d\tilde{y}^i_{j_1} &= (D_\beta {G^i}_{j_1}) dx^\beta. \end{aligned} \tag{1.36}$$

substituting (1.36) into (1.35), and solving for $\tilde{y}^i_{j_1 j_2}$ using the same procedure as in the last section:

$$\tilde{y}^i_{j_1 j_2} = D_\beta {G^i}_{\{j_1\}} (D_{j_2} F^\beta)^{-1} = {G^i}_{j_1 j_2}[\mathbf{x}, \mathbf{y}, \mathbf{y_1}, \mathbf{y_2}, s], \tag{1.37}$$

Where $\mathbf{y_2}$ refers to the vector of all possible second partial derivatives.

Similarly the transformation (1.24) is extended to the pth derivative by utilizing the contact condition,

$$\begin{aligned} d(\tilde{y}^i) - \tilde{y}^i_\alpha &= 0, \\ d(\tilde{y}^i_{j_1}) - \tilde{y}^i_{j_1\alpha} d\tilde{x}^\alpha &= 0, \\ &\vdots \\ d(\tilde{y}^i_{j_1 j_2 \ldots j_{p-1}}) - \tilde{y}^i_{j_1 j_2 \ldots j_p} d\tilde{x}^\alpha &= 0. \end{aligned} \tag{1.38}$$

at successive orders. the pth extended finite group is

$$
\begin{aligned}
\tilde{x}^j &= F^j[\mathbf{x}, \mathbf{y}, s], \quad j = 1, \ldots, n, \\
\tilde{y}^i &= G^j[\mathbf{x}, \mathbf{y}, s], \quad j = 1, \ldots, m \\
\tilde{y}^i_{j_1} &= G^j_{j_1}[\mathbf{x}, \mathbf{y}, \mathbf{y_1}, s], \\
&\vdots \\
\tilde{y}^i_{j_1 j_2 \ldots j_p} &= G^j_{j_1 j_2 \ldots j_p}[\mathbf{x}, \mathbf{y}, \mathbf{y_1}, \ldots, \mathbf{y_p}, s]
\end{aligned}
\tag{1.39}
$$

where

$$
{G^i}_{j_1 j_2 \ldots j_p}[\mathbf{x}, \mathbf{y}, \mathbf{y_1}, \ldots, \mathbf{y_p}, s] = D_\beta {G^i}_{\{j_1 j_2 \ldots j_{p-1}\}}(D_{j_p} F^\beta)^{-1}. \tag{1.40}
$$

As usual, $s$ is the group parameter which defines the mapping from the source space

$$
(\mathbf{x}, \mathbf{y}[\mathbf{x}], \mathbf{y_1}[\mathbf{x}], \mathbf{y_2}[\mathbf{x}], \ldots, \mathbf{y_p}[\mathbf{x}])
$$

to the target space

$$
(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}[\tilde{\mathbf{x}}], \tilde{\mathbf{y}}_\mathbf{1}[\tilde{\mathbf{x}}], \tilde{\mathbf{y}}_\mathbf{2}[\tilde{\mathbf{x}}], \ldots, \tilde{\mathbf{y}}_\mathbf{p}[\tilde{\mathbf{x}}]).
$$

**Infinitesimal Transformation of First Partial Derivatives**

The infinitesimal transformation corresponding to (1.24) is

$$
T^s = \begin{cases} \tilde{x}^j = x^j + s\xi^j[\mathbf{x}, \mathbf{y}], \quad j = 1, \ldots, n \\ \tilde{y}^i = y^i + s\eta^i[\mathbf{x}, \mathbf{y}], \quad i = 1, \ldots, m \end{cases} \tag{1.41}
$$

generated by expanding (1.24) in a Taylor series about $s = 0$. The infinitesimals are

$$\xi^j[\mathbf{x}, \mathbf{y}] = \left(\frac{\partial F^j}{\partial s}\right)_{s=0}, \qquad \eta^i[\mathbf{x}, \mathbf{y}] = \left(\frac{\partial G^i}{\partial s}\right)_{s=0}. \tag{1.42}$$

Substituting $F^\beta = x^\beta + s\xi^\beta$ and $G^i = y^i + s\eta^i$ into (1.32) produces

$$\tilde{y}^i_j = (D_\beta(y^i + s\eta^i))(D_j(x^\beta + s\xi^\beta))^{-1}. \tag{1.43}$$

The group parameter $s$ is assumed to be small, and so the matrix inverse can be approximated using

$$(\delta^\beta_j + sD_j\xi^\beta)^{-1} \approx \delta^\beta_j - s(D_j\xi^\beta). \tag{1.44}$$

To derive (1.44) we have used the general exponential form of a matrix. Let $A^\beta_j = D_j\xi^\beta$; then the matrix $exp(sA^\beta_j) \approx \delta^\beta_j + sA^\beta_j + o(s^2) + \ldots$ has the inverse $exp(-sA^\beta_j) \approx (\delta^\beta_j + sA^\beta_j + o(s^2) + \ldots)^{-1} \approx \delta^\beta_j - sA^\beta_j + o(s^2) - \ldots$. Using this result, Eq. (1.43) becomes

$$\tilde{y}^i_j = (y^i_\beta + sD_\beta\eta^i)(\delta^\beta_j - sD_j\xi^\beta). \tag{1.45}$$

Retaining only lowest-order terms in s, the infinitesimal form of the transformation of first partial derivatives is now determined to be

$$\tilde{y}^i_j = y^i_j + s(D_j\eta^i - y^i_\beta D_j\xi^\beta), \tag{1.46}$$

and the once extended infinitesimal group is

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\mathbf{x}, \mathbf{y}], \qquad j = 1, \ldots, n \\
\tilde{y}^i &= y^i + s\eta^i[\mathbf{x}, \mathbf{y}], \qquad i = 1, \ldots, m \\
\tilde{y}^i_j &= y^i_j + s\eta^i_{\{j\}}[\mathbf{x}, \mathbf{y}, \mathbf{y_1}],
\end{aligned} \tag{1.47}$$

where

$$\eta^i_{\{j\}}[\mathbf{x}, \mathbf{y}, \mathbf{y_1}] = D_j\eta^i - y^i_\beta\xi^\beta. \tag{1.48}$$

**Infinitesimal Transformation of Second and Higher Partial Derivatives**

The transformation of second partial derivatives is generated in the same way. Substitute $F^\beta = x^\beta + s\xi^\beta$ and $G^i_{j_1} = y^i_{j_1} + s\eta^i_{j_1}$ into the finite transformation (1.36):

$$\tilde{y}^i_{j_1j_2} = (D_\beta(y^i_{j_1} + s\eta^i_{\{j_1\}}))(D_{j_2}(x^\beta + s\xi^\beta))^{-1}. \tag{1.49}$$

Carrying out the differentiation indicated in (1.49):

$$\tilde{y}^i_{j_1j_2} = (y^i_{j_1\beta} + sD_\beta\eta^i_{\{j_1\}})(\delta^\beta_{j_2} + sD_{j_2}\xi^\beta)^{-1}. \tag{1.50}$$

we approximate the inverse using (1.44) and retaining only the lowest-order term in $s$ to produce the infinitesimal transformation of second partial derivatives,

$$\tilde{y^i}_{j_1j_2} = y^i_{j_1j_2} + s(D_{j_2}\eta^i_{\{j_1\}} - y^i_{j_1\beta}D_{j_2}\xi^\beta). \tag{1.51}$$

The twice extended group is

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\mathbf{x},\mathbf{y}], \quad j = 1,\ldots,n \\
\tilde{y}^i &= y^i + s\eta^i[\mathbf{x},\mathbf{y}], \quad i = 1,\ldots,m \\
\tilde{y}^i_j &= y^i_j + s\eta^i_{\{j_1\}}[\mathbf{x},\mathbf{y},\mathbf{y_1}], \\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\mathbf{x},\mathbf{y},\mathbf{y_1},\mathbf{y_2}],
\end{aligned} \tag{1.52}$$

where

$$\eta^i_{\{j_1 j_2\}} = D_{j_2}\eta^i_{j_1} - y^i_{j_1\beta} D_{j_2}\xi^\beta. \tag{1.53}$$

Where, the total derivative operator acting on the first extention is

$$D_{j_2}\eta^i_{\{j_1\}} = \frac{\partial \eta^i_{\{j_1\}}}{\partial x^{j_2}} + y^\alpha_{j_2}\frac{\partial \eta^i_{\{j_1\}}}{\partial y^\alpha} + y^\alpha_{\beta j_2}\frac{\partial \eta^i_{\{j_1\}}}{\partial y^\alpha_\beta}. \tag{1.54}$$

For pth derivatives the extended infinitesimal transformation is

$$\begin{aligned}
\tilde{x}^j &= x^j + s\xi^j[\mathbf{x},\mathbf{y}], \quad j = 1,\ldots,n \\
\tilde{y}^i &= y^i + s\eta^i[\mathbf{x},\mathbf{y}], \quad i = 1,\ldots,m \\
\tilde{y}^i_j &= y^i_j + s\eta^i_{\{j_1\}}[\mathbf{x},\mathbf{y},\mathbf{y_1}], \\
\tilde{y}^i_{j_1 j_2} &= y^i_{j_1 j_2} + s\eta^i_{\{j_1 j_2\}}[\mathbf{x},\mathbf{y},\mathbf{y_1},\mathbf{y_2}], \\
&\vdots \\
\tilde{y}^i_{j_1 j_2 \ldots j_p} &= y^i_{j_1 j_2 \ldots j_p} + s\eta^i_{\{j_1 j_2 \ldots j_p\}}[\mathbf{x},\mathbf{y},\mathbf{y_1},\mathbf{y_2},\ldots,\mathbf{y_p}],
\end{aligned} \tag{1.55}$$

where

$$\eta^i_{\{j_1 j_2 \ldots j_p\}} = D_{j_p}\eta^i_{\{j_1 j_2 \ldots j_{p-1}\}} - y^i_{j_1 j_2 \ldots j_{p-1}\alpha} D_{j_p}\xi^\alpha. \tag{1.56}$$

The total differentiation operator appearing in (1.56) is

$$D_{j_p}() = \frac{\partial()}{\partial x^{j_p}} + y^i_{j_p}\frac{\partial()}{\partial y^i} + y^i_{j_1 j_p}\frac{\partial()}{\partial y^i_{j_1}} + y^i_{j_1 j_2 j_p}\frac{\partial()}{\partial y^i_{j_1 j_2}} + \cdots + y^i_{j_1 j_2 \dots j_{p-1} j_p}\frac{\partial()}{\partial y^i_{j_1 j_2 \dots j_{p-1}}}. \tag{1.57}$$

## 1.3 Approximate Transformation Groups and Symmetries

### 1.3.1 Approximate Transformation Groups

#### Notation and Definitions

Before continuing we need to present some definitions and theorems from the Ref.[16]. If a function $f(x,\epsilon)$ satisfies the condition

$$\lim_{x\to 0}\frac{f(x,\epsilon)}{\epsilon^p} = A \quad , where \;\; A \;\; is \;\; a \;\; constant$$

then $f(x,\epsilon) = O(\epsilon^p)$ and $f$ is said to be of order $O(\epsilon^p)$, $x \to 0$.

If

$$\lim_{x\to 0}\frac{f(x,\epsilon)}{\epsilon^p} = 0,$$

then $f$ is said to be of order $o(\epsilon^p)$.

If $f(x,\epsilon) - g(x,\epsilon) = O(\epsilon^p)$ or briefly $f \sim g$ when there is no ambiguity. Given $f(x,\epsilon)$, let

$$f_0(x) + \epsilon f_1(x) + \cdots + \epsilon^p f_p(x)$$

be the approximate polynomial of degree $p$ in $\epsilon$ obtained via the Taylor series expansion of $f(x, \epsilon)$ in power of $\epsilon$ about $\epsilon = 0$. Then any function $g \approx f$ has the form $g(x) = f_0(x) + \epsilon f_1(x) + \cdots + \epsilon^p f_p(x) + o(\epsilon^p)$. consequently the function

$$f_0(x) + \epsilon f_1(x) + \cdots + \epsilon^p f_p(x)$$

is called a *canonical representative* of the equivalence class of functions containing $f$. So, the one-parameter family G of approximate transformations, defined as follows

$$\bar{x}^i = f_0^i(x, a) + \epsilon f_1^i(x, a) + \cdots + \epsilon^p f_p^i(x, a) \quad i = 1, \cdots, n \tag{1.58}$$

of points $x = (x^1, \cdots, x^n) \in R^n$ into $\bar{x} = (\bar{x}^1, \cdots, \bar{x}^n) \in R^n$ as the class of invertible transformations

$$\bar{x} = f(x, a, \epsilon), \tag{1.59}$$

with vector-functions $f = (f^1, \cdots, f^n)$ such that

$$f^i(x, a, \epsilon) \approx f_0^i(x, a) + \epsilon f_1^i(x, a) + \cdots + \epsilon^p f_p^i(x, a) \quad i = 1, \cdots, n.$$

Here, $a$ is a real parameter, and the following condition is imposed:

$$f(x, 0, \epsilon) \approx x.$$

**Definition 1.4** *The set of transformations (1.58) is called a one-parameter approximate transformation group if*

$$f(f(x,a,\epsilon),b,\epsilon)\approx f(x,a+b,\epsilon)$$

*for all transformations(1.59).*

*The generator of an approximate transformation group G given by (1.59) is the class of first-order linear differential operartors*

$$X=\xi^i(x,\epsilon)\frac{\partial}{\partial x^i}$$

*such that*

$$\xi^i(x)\approx\xi_0^i(x)+\epsilon\xi_1^i(x)+\cdots+\epsilon^p\xi_p^i(x),$$

*where the vector fields* $\xi_0,\xi_1,\cdots,\xi_p$ *are given by*

$$\xi_v^i(x)=\frac{\partial f_v^i(x,a)}{\partial a}\mid_{a=0},\quad v=0,\cdots,p\quad i=1,\cdots,n.$$

*In what follows, an approximate group generator is*

$$X\approx(\xi_0^i(x)+\epsilon\xi_1^i(x)+\cdots+\epsilon^p\xi_p^i(x))\frac{\partial}{\partial x^i}$$

*In theoretical discussions, approximate equalities are considered with an error* $o(\epsilon^p)$ *of an arbitrary order* $p\geq 1$. *However, in most of the applications the theory is simplified by letting* $p=1$.

**Approximate Lie Equations**

The one-parameter approximate transformation groups in the first order of precision, i.e., Eqs.(1.58) of the form

$$\bar{x}^i \approx f_0^i(x,a) + \epsilon f_1^i(x,a) \quad i = 1, \cdots, n. \tag{1.60}$$

let

$$X = X_0 + \epsilon X_1 \tag{1.61}$$

be a given approximate operator, where

$$X_0 = \xi_0^i(x)\tfrac{\partial}{\partial x^i}, \quad X_1 = \xi_1^i(x)\tfrac{\partial}{\partial x^i}$$

recall that this means $X_0 = \sum_{i=1}^n \xi_0^i(x)\frac{\partial}{\partial x^i}$ and $X_1 = \sum_{i=1}^n \xi_1^i(x)\frac{\partial}{\partial x^i}$, respectively. The corresponding approximate transformation (1.60) of points $x$ into points $\bar{x} = \bar{x}_0 + \epsilon \bar{x}_1$ with the coordinates

$$\bar{x}^i = \bar{x}_0^i + \epsilon \bar{x}_1^i \tag{1.62}$$

where

$$\bar{x}_0^i = f_0^i(x,a), \quad \bar{x}_1^i = f_1^i(x,a),$$

is determined by the following equations:

$$\frac{dx_0^i}{da} = \xi_0^i(\bar{x}_0), \quad \bar{x}_0^i \mid_{a=0}= x^i, \quad i = 1, \cdots, n, \tag{1.63}$$

$$\frac{dx_1^i}{da} = \sum_{k=1}^{n} \frac{\partial \xi_0^i(x)}{\partial x^k} \mid_{x=\bar{x}_0} \bar{x}_1^k + \xi_1^i(\bar{x}_0), \quad \bar{x}_1^i \mid_{a=0}= 0 \quad i = 1, \cdots, n. \tag{1.64}$$

Eqs.(1.63)-(1.64) are called the approximate Lie equations.

**Example 3** *Let $n = 2$ and let*

$$X = (1 + \epsilon x^2)\tfrac{\partial}{\partial x} + \epsilon xy\tfrac{\partial}{\partial y}$$

*here $\xi_0(x, y) = (1, 0)$, $\xi_1(x, y) = (x^2, xy)$, and Eqs.(1.63)-(1.64) are written as*

$$\frac{d\bar{x}_0}{da} = 1, \qquad \frac{d\bar{y}_0}{da} = 0, \qquad \bar{x}_0 \mid_{a=0}= x, \quad \bar{y}_0 \mid_{a=0}= y,$$

$$\frac{d\bar{x}_1}{da} = \bar{x}_0^2, \qquad \frac{d\bar{y}_1}{da} = \bar{x}_0\bar{y}_0, \qquad \bar{x}_1 \mid_{a=0}= 0, \quad \bar{y}_1 \mid_{a=0}= 0.$$

*The integration gives the following approximate transformation group:*

$$\bar{x} \approx x + a + \epsilon(ax^2 + a^2x + \frac{a^3}{3}), \qquad \bar{y} \approx y + \epsilon(axy + \frac{a^2}{2}y).$$

**Approximate Exponential Map**

**Theorem 1.2** *Given an operator*

$$X = X_0 + \epsilon X_1,$$

*with a small parameter $\epsilon$ ,where*

$$X_0 = \xi_0^i \frac{\partial}{\partial x^i}, \quad X_1 = \xi_1^i \frac{\partial}{\partial x^i}$$

*the corresponding approximate group transformation*

$$\bar{x}^i = \bar{x}_0^i + \epsilon \bar{x}_1^i, \quad i = 1, \cdots, n,$$

*are determined by the following equation :*

$$\bar{x}_0^i = e^{aX_0}(x^i), \quad \bar{x}_1^i = \ll aX_0, aX_1 \gg (\bar{x}_0^i), \quad i = 1, \cdots, n,$$

*where*

$$e^{aX_0} = 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots .$$

*and*

$$\ll aX_0, aX_1 \gg = aX_1 + \frac{a^2}{2!}[X_0, X_1] + \frac{a^3}{3!}[X_0, [X_0, X_1]] + \cdots .$$

*In other words, the approximate operator $X = X_0 + \epsilon X_1$ generates the one-parameter approximate transformation group given by the following approximate exponential map:*

$$\bar{x}^i = (1 + \epsilon \ll aX_0, aX_1 \gg) e^{aX_0}(x^i), \quad i = 1, \cdots, n.$$

**Example 4** *Let us apply the previous Theorem to the operator*

$$X = (1 + \epsilon x)\frac{\partial}{\partial x}.$$

*such that*

$$X_0 = \frac{\partial}{\partial x}, \quad X_1 = x\frac{\partial}{\partial x}$$

*therefore*

$$X_0(x) = 1, \quad X_0^2(x) = X_0^3(x) = \cdots = 0,$$

*and*

$$[X_0, X_1] = \frac{\partial}{\partial x} = X_0,$$

$$[X_0, [X_0, X_1]] = [X_0, X_0] = 0, \cdots$$

*consequently,*

$$\bar{x}_0 = e^{aX_0}(x) = x + a,$$

*and*

$$\ll aX_0, aX_1 \gg (\bar{x}_0) = (ax + \frac{a^2}{2!})\frac{\partial}{\partial x}(x + a) = ax + \frac{a^2}{2!}$$

*hence*

$$\bar{x} \approx x + a + \epsilon(ax + \frac{a^2}{2}).$$

### 1.3.2 Approximate Symmetries

Approximate transformation groups admitted by differential equations with small parameter $\epsilon$. The method of the Approximation will be illustrated.

**Definitions of approximate Symmetries**

**Definition 1.5** *Let $G$ be a one-parameter approximate transformation group:*

$$\tilde{z}^i \approx f_0^i(z,a) + \epsilon f_1^i(z,a), \qquad i = 1, \cdots, n \tag{1.65}$$

*an approximate equation*

$$F(z,\epsilon) = F_0(z) + \epsilon F_1(z) \approx 0 \tag{1.66}$$

*is said to be approximately invariant with respect to $G$,or admits $G$ if*

$$F(z,\epsilon) \approx F(f(z,a,\epsilon),\epsilon) \approx o(\epsilon) \tag{1.67}$$

*whenever $z = (z^1, z^1, \ldots, z^n)$ satisfies Eq.(1.66).*

*If $z = (x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)})$, then Eq.(1.66) becomes an approximate differential equation of order $k$, and $G$ becomes an approximate symmetry group of the differential equation.*

**Determining Equation and Stable Symmetries**

**Theorem 1.3** *Eq.* (1.66) *is approximatly invariant under the approximate transformation group* (1.65) *with the generator*

$$X = X^0 + \epsilon X^1 \equiv \xi_0^i(z)\frac{\partial}{\partial z^i} + \epsilon\xi_1^i\frac{\partial}{\partial z^i} \tag{1.68}$$

*if and only if*

$$[XF(z,\epsilon)]_{F\approx 0} = o(\epsilon) \tag{1.69}$$

*or*

$$[X^0F_0(z) + \epsilon(X^1F_0(z) + X^0F_1(z))]_{(1.68)} = o(\epsilon) \tag{1.70}$$

*The operator given by Eq.*(1.68) *that satisfies Eq.*(1.70) *is called and infinitesimal approximate symmetry, or an approximate operator admitted by Eq.*(1.65). *Accordingly, Eq.*(1.70) *is termed as the determining equation for approximate symmetries.*

**Remark 1** *The determining equation* (1.70) *can be written as follows:*

$$X^0F_0(z) = \lambda F_0(z), \tag{1.71}$$

$$X^1F_0(z) + X^0F_1(z) = \lambda F_1(z). \tag{1.72}$$

*The factor $\lambda$ is determined by Eq.*(1.71) *and then substituted into Eq.*(1.72). *The latter equation must hold for all solutions of $F_0(z) = 0$.*

**Theorem 1.4** *if Eq.*(1.66) *admits an approximate transfromation group with the generator* $X = X^0 + \epsilon X^1$, *where* $X^0 \neq 0$, *then the operator*

$$X^0 \equiv \xi_0^i(z) \frac{\partial}{\partial z^i} \tag{1.73}$$

*is an exact symmetry of the equation*

$$F_0(z) = 0. \tag{1.74}$$

**Definition 1.6** *Eqs.*(1.74) *and* (1.66) *are termed an unperturbed equation and perturbed equation, respectively. Under the conditions of theorem 1.4, the operator* $X^0$ *is called a stable symmetry of the unperurbed equation*(1.74)*. The corresponding approximate symmetry generator* $X = X^0 + \epsilon X^1$ *for the perturbed equation* (1.73) *is called a deformation of the infinitesimal* $X^0$ *of Eq.* (1.74) *caused by the perturbation* $\epsilon F_1(z)$.

*In particular, if the most general symmetry Lie algebra of Eq.* (1.74) *is stable, we say that the perturbed equation* (1.66) *inherits the symmetries of the unperturbed equation.*

### Calculation of Approximate Symmetries

Remark 1 and Theorem 1.4 provide an infinitesimal method for calculating approximate symmetries (1.68) for differential equations with a small parameter. Implementation of the method requires the following three steps.

**1st Step:** Calculation of the exact symmetries $X^0$ of the unperturbed equation (1.74), by solving the determining equation

$$X^0 F_0(z)|_{F_0(z)=0} = 0. \tag{1.75}$$

**2nd Step:** Determining of the auxiliary function $H$ by virtue of Eqs. (1.72),(1.71) and(1.66), i.e., by the equation

$$H = \frac{1}{\epsilon}[X^0(F_0(z) + \epsilon F_1(z))|_{F_0(z)+\epsilon F_1(z)=0}] \tag{1.76}$$

with known $X^0$ and $F_0(z)$.

**3rd Step:** Calculation of the operator $X^1$ by solving the determining equation for deformations:

$$X^1 F_1(z)|_{F_0(z)=0} + H = 0. \tag{1.77}$$

Note that Eq.(1.77), unlike the determining equation (1.75) for exact symmetries, is inhomogeneous.

### 1.3.3 Approximate Lie Algbras

Approximate symmetries from an approximate Lie algebras [13].

**Definition 1.7** *An approximate commutator of the approximate operators* $X^1$ *and* $X^2$ *is an approximate operator denoted by* $[X_1, X_2]$ *and is given by*

$$[X_1, X_2] \approx X_1X_2 - X_2X_1.$$

*The approximate commutator satisfies the usual properties, namely:*

- *Linearity:* $[aX_1 + bX_2, X_3] \approx a[X_1, X_3] + b[X_2, X_3], \quad a, b = const,$
- *skew-symmetry:* $[X_1, X_2] \approx -[X_2, X_1],$
- *Jacobi identity:* $[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] \approx 0.$

**Definition 1.8** *A vector space $L$ of approximate operators is called an approximate Lie algebra of operator if it is closed (in approximation of the given order $p$) under the approximate commutator,i.e., if*

$$[X_1, X_2] \in L$$

*for any $X_1, X_2 \in L$. Here the approximate commutator $[X_1, X_2]$ is calculated to the precision indicated.*

**Example 5** *Consider the approximate (up to $o(\epsilon)$) operators*

$$X_1 = \tfrac{\partial}{\partial x} + \epsilon x \tfrac{\partial}{\partial y}, \qquad X_2 = \tfrac{\partial}{\partial y} + \epsilon y \tfrac{\partial}{\partial x}.$$

*Their linear span is not a Lie algebra in the usual(exact) sense. For instance, the (exact) commutator*

$$[X_1, X_2] = \epsilon^2 (x \tfrac{\partial}{\partial x} - y \tfrac{\partial}{\partial y})$$

*is not a linear combination of the above operators.*

*However, these operators span an approximate Lie algebra in the first-order of precision.*

## 1.4 Description of our Undertaking

We summarize the main objective to be achieved in this study

- To study different approaches for approximate symmetries.
- Compare these different approaches by applying to some non-linear partial differential equations.
- Apply a suitable approach to find the solution to a model of non-linear Pde using approximate Lie symmetries.

# CHAPTER 2

# APPROXIMATE LIE SYMMETRY METHODS

## 2.1 Introduction

Lie group theory provides a systematic way of finding exact solutions to differential equations. If the problem involves a small parameter, then an approximate solution instead of an exact solution can be sought. We describe four methods in which a combination of Lie symmetries and perturbation theory is used to find approximate Lie symmetries and approximate solutions.

**METHOD I** is due to Baikov,Gazizov and Ibragimov [4],[2]. In this method an approximate generator is calculated to find solutions. In this method the dependent variable is not expanded in a perturbation series. One may refer the reader to section 1.3.2 for more details about this method.

**METHOD II** is due to Fushchich and Shtelen [12] and later followed by Euller et al [10], Euler and Euler [11]. In this method the dependent variables are expanded in a perturbation series first as is done in the usual perturbation analysis. The approximate symmetry of the original equation is defined to be the exact symmetry of the coupled equations.

Consider the general *mth* order nonlinear evolution equation

$$E = E(x, t, u, u_1, u_2, \ldots, u_m, u_t; \epsilon) = 0, \tag{2.1}$$

Where $u_t = \partial u/\partial t, u_k = \partial^k u/\partial x^k, 1 \leqslant k \leqslant m$ and $\epsilon$ is a small parameter, $E$ is a smooth function of the indicated variables.

Expanding the dependent variable in the small parameter yields

$$u = u_0 + \epsilon u_1 + \ldots, \qquad 0 < \epsilon < 1 \tag{2.2}$$

then inserting expansion (2.2) into the original Eq.(2.1) and separating at each order of the perturbed parameter, one has

$$Order \;\; \epsilon^0 : \quad E_0 = 0, \qquad Order \;\; \epsilon^1 : \quad E_1 = 0, \tag{2.3}$$

hence the exact symmetry of system (2.3) is the approximate symmetry of the original Eq.(2.1).

**METHOD III** is actually a modification of the second method and consistent with the perturbation theory. The approximate symmetry of the original equations as the exact symmetry of the non-homogeneous linear equation and the "couple-equations" assumption removed. Consider a nonlinear equation with a small parameter $\epsilon$ as follows

$$L(u) + \epsilon N(u) = 0 \tag{2.4}$$

Where $L$ is an arbitrary linear and $N$ is an arbitrary nonlinear differential operators. $x_i$ are the independent variables and $u$ is the dependent variable such that $u = u(x_1, x_2, \ldots, x_n)$. Expanding $u$ in a perturbation series

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 \ldots \tag{2.5}$$

and substituting into the original equation, one has

$$L(u_0) = 0 = h_0(x_1, x_2, \ldots, x_n)$$

$$L(u_1) = -N(u_0) = h_1(x_1, x_2, \ldots, x_n)$$

$$L(u_2) = -N(u_0, u_1) = h_2(x_1, x_2, \ldots, x_n)$$

Note that the left-hand side of all the equations are the same and the right-hand sides of the equations can be considered as arbitrary functions of the independent variables which are to be determined sequentially starting from the first equation.

**Definition 1** *The approximate symmetry of the nonlinear equation* (2.4) *is the exact symmetry of the following linear nonhomogeneous equation*

$$L(u) = h(x_1, x_2, \ldots, x_n) \tag{2.6}$$

*with $h$ considered as an arbitrary function.*

*Since $h_i$ are in fact known function, substituting these forms to the general symmetry of Equation* (2.6) *one obtains the specific symmetries at each level of approximation.*

**METHOD IV** is due to Zhi-Yong, Xue-Lin and Yu-Fu [28]. In this method both the independent variable and the dependent variables are simultaneously expanded in the small parameter series to obtain more affluent approximate symmetries than those achieved by expanding the dependent variable in the small parameter series only.

The dependent variable expands as Eq.(2.2) and the independent variable is expanded to

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tilde{t}} + \epsilon \frac{\partial}{\partial s}, \quad \tilde{t} = t, \quad s = \epsilon t, \quad 0 < \epsilon \leqslant 1. \tag{2.7}$$

Inserting expansions (2.2) and (2.7) into the original Eq.(2.1) and separating at each order of the small parameter, then we have the following equations with repect to different orders $\epsilon^0, \epsilon^1$ and $\epsilon^2$

$$Order \ \ \epsilon^0: \quad \tilde{E}_0 = 0, \qquad Order \ \ \epsilon^1: \quad \tilde{E}_1 = 0, \qquad Order \ \ \epsilon^2: \quad \tilde{E}_2 = 0, \tag{2.8}$$

so the aproximate symmetry of the original Eq.(2.1) is defined as the exact symmetry of system (2.8).

Due to the increase of the number of equations, from two to three, we easily find that calculation of this method increases enormously and the problem can only be handled by a symbolic computer package.

## 2.2 Approximate Symmetries of Perturbed Non-linear Wave Equation by Method I

Let us consider perturbed equation [16, p. 56]

$$F_0(z) + \epsilon F_1(z) = u_{tt} - (u^2 u_x)_x + \epsilon u_t = 0. \tag{2.9}$$

and write the approximate group generator in the form

$$X = X^0 + \epsilon X^1 \equiv (\tau_0 + \epsilon\tau_1)\frac{\partial}{\partial t} + (\xi_0 + \epsilon\xi_1)\frac{\partial}{\partial x} + (\eta_0 + \epsilon\eta_1)\frac{\partial}{\partial u}, \tag{2.10}$$

where $\xi_v$, $\tau_v$ and $\eta_v$ $(v = 0, 1)$ are unknown functions of $t, x$, and u.

**1st Step.** Solving the following determining equation for the exact symmetries $X^0$ of the unperturbed equation, $F_0(z) = 0$

$$X^0 F_0(z)|_{F_0(z)=0} = 0. \tag{2.11}$$

The infinitesimal generator for the unperturbed equation is a vector field on a three dimensional space, as there are two independent variables and one dependent variable,therefore,

$$X^0 = \tau_0\frac{\partial}{\partial t} + \xi_0\frac{\partial}{\partial x} + \eta_0\frac{\partial}{\partial u} \tag{2.12}$$

We need to prolong the symmetry generator (2.12) to second order. Thus the prolongation of the infinitesimal symmetry generator is as follows

$$X^{0^{[2]}} = X^0 + \eta_o^t \frac{\partial}{\partial u_t} + \eta_o^x \frac{\partial}{\partial u_x} + \eta_o^{tt} \frac{\partial}{\partial u_{tt}} + \eta_o^{xt} \frac{\partial}{\partial u_{xt}} + \eta_o^{xx} \frac{\partial}{\partial u_{xx}} \tag{2.13}$$

Symmetry criterion (2.11) for the unperturbed partial differential equation gives the relation

$$X^{0^{[2]}}(u_{tt} - (u^2 u_x)_x)|_{u_{tt}=(u^2 u_x)_x} = 0. \tag{2.14}$$

By comparing coefficients of $u_x$, ${u_x}^2, \cdots$ we obtain the following system of ten determining equations.

e1: $\xi_{0u} = 0$

e2: $\xi_{0t} = 0$

e3: $\tau_{0u} = o$

e4: $\tau_{0x} = 0$

e5: $\eta_{0uu} = 0$

e6: $2\eta_0 + 2u\eta_{0u} - 4u\xi_{0x} + 4u\tau_{0t} = 0$

e7: $2u^2\eta_{0xu} - u\xi_{0xx} + 4u\eta_{0x} = 0$

e8: $-2\eta_{0tu} + \tau_{0tt} = 0$

e9: $2u\eta_0 + 2u^2\xi_{0x} + 2u^2\tau_{0t} = 0$

e10: $u^2\eta_{0xx} - \eta_{0tt} = 0$

Differentiating (e10) twice with respect to u and using (e5) implies

e11: $\eta_0 = \phi_1(t)u + \phi_2(t)$ ,

from (e6), differentiating with respect to u and x respectively, gives

e12: $\xi_{0xx} = 0.$

Using (e1),(e2) and (e12), we obtain

e13: $\xi_0 = \alpha_0 + \alpha_1 x.$

Using $(e9)_{uu}$ , we obtain

e14: $eta_{0u} - \xi_{0x} + \tau_{0t} = 0$

By (e13) and (e11), e14 reduces to

e15: $\tau_0 = -\int \phi_1(t), dx + \alpha_1 t + \alpha_2.$

Using (e15) in $(e9)_u$, we obtain $\phi_2(t) = 0$, substituting $(e15)_{tt}$ into (e8) gives $\phi_1(t) = 0$, implies $\phi_1(t) = \alpha_3$.

Hence,

$$\begin{aligned} \xi_0 &= \alpha_0 + \alpha_1 x \\ \tau_0 &= \alpha_3 t + \alpha_1 t + \alpha_2 \\ \eta_0 &= -\alpha_3 u \end{aligned} \tag{2.15}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are arbitrary constants. Hence,

$$X^0 = (\alpha_0 + \alpha_1 x)\frac{\partial}{\partial x} + (\alpha_3 t + \alpha_1 t + \alpha_2)\frac{\partial}{\partial t} - \alpha_3 u \frac{\partial}{\partial u}. \tag{2.16}$$

**2nd Step.** Determination of auxilary function H.for this, we consider

$$H = \frac{1}{\epsilon}[X^{0\{2\}}[f_0(z) + \epsilon F_1(z)] \mid_{F_0(z)+\epsilon F_1(z)=0}, \tag{2.17}$$

or

$$H = \frac{1}{\epsilon}[X^{0\{2\}}(u_{tt} - 2uu_x^2 - u^2u_{xx} + \epsilon u_t)]|_{\{(u_{tt}-2uu_x^2-u^2u_{xx}+\epsilon u_t)\}} \tag{2.18}$$

where $X^{0\{2\}}$ is the second prolongation of $X^0$,implies

$$H = \frac{1}{\epsilon}[\eta(-2u_x^2 - 2uu_{xx}) + \eta^x(-4uu_x^2) + \eta^t(\epsilon) + \eta^{xx}(-u^2) + \eta^{tt}]|_{\{(u_{tt}-2uu_x^2-u^2u_{xx}+\epsilon u_t)\}}. \tag{2.19}$$

So that

$$\eta = -\alpha_3 u$$

$$\eta^x = -\alpha_3 u_x$$

$$\eta^t = \alpha_4 u_t - \alpha_3 u_t$$

$$\eta^{tt} = \alpha_4 u_{tt} - 2\alpha_3 u_{tt}$$

$$\eta^{xx} = -\alpha_4 u_{xx} - 2\alpha_3 u_{xx}$$

Substitute $\eta, \eta^x, \eta^t, \eta^{tt}, \eta^{xx}$ and $u_{tt} = 2uu_x^2 + u^2u_{xx} - \epsilon u_t$ into (2.19), hence

$$H = \alpha_3 u_t. \tag{2.20}$$

**3rd Step:** The determining equation (1.77) for deformations is written as

$$X^{1\{2\}}(u_{tt} - u^2u_{xx} - 2uu_x^2)|_{u_{tt}=u^2u_{xx}+2uu_x^2} + H = 0, \tag{2.21}$$

Where $X^{1\{2\}}$ denotes the second prolongation of the operator

$$X^1 = \tau_1 \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial u} \tag{2.22}$$

Then it follows that the system of the determining equations for (2.21) is given by

e1: $\xi_{1u} = 0$

e2: $\xi_{1t} = 0$

e3: $\tau_{1u} = o$

e4: $\tau_{1x} = 0$

e5: $\eta_{1uu} = 0$

e6: $2\eta_1 + 2u\eta_{1u} - 4u\xi_{1x} + 4u\tau_{1t} = 0$

e7: $2u^2\eta_{1xu} - u\xi_{1xx} + 4u\eta_{1x} = 0$

e8: $-2\eta_{1tu} + \tau_{1tt} - \alpha_3 = 0$

e9: $2u\eta_1 + 2u^2\xi_{1x} + 2u^2\tau_{1t} = 0$

e10: $u^2\eta_{1xx} - \eta_{1tt} = 0$

From (e1)and (e2) implies $\xi_1 = A(x)$, from (e3) and (e4) implies $\tau_1 = B(t)$ and from (e5) implies $\eta_1 = \psi_1(x,t)u + \psi_2(x,t)$.

We differentiate (e7) to get

e11:$4\eta_1 x + 4u\eta_{1xu} - \xi_{1xx} = 0$.

From (e6), differentiating with respect to $u, x$ respectively gives

e12: $\eta_{1ux} - \xi_{1xx} = 0.$

Using (e12) in $(e6)_u$ gives,

$\eta_{1xu} = 0$, which implies $\eta_1 = \psi_1(t)u + \psi_2(t), \quad \xi_1 = \beta_0 + \beta_1 x.$

From (e9) differentiate with respect to u twice to obtain

e14: $4\eta_u - 4\xi_x + 4\tau_t = 0$

Substituting $\xi_1 = \beta_0 + \beta_1 x$ in (e14) gives

e15: $\tau_1 = \beta_1 t - \int \psi_1(t), dt + \beta_2$

Using (e15) in (e16),(e8) leads to

$\psi_1(t) = \beta_3 - \frac{1}{3}\alpha_3$. Hence

$$\begin{aligned} \tau_1 &= \beta_1 + \beta_3 t + \tfrac{1}{6}\alpha_3 t^2 \\ \xi_1 &= \beta_2 + (\beta_3 + \beta_4)x \\ \eta_1 &= (\beta_4 - \tfrac{1}{3}\alpha_3 u)u \end{aligned} \tag{2.23}$$

Substituting (2.15) and (2.23) into (2.10), we obtain the following approximate symmetries for (2.9):

$$\begin{gathered} X_1 = \tfrac{\partial}{\partial t}, \quad X_2 = \tfrac{\partial}{\partial x}, \quad X_3 = t\tfrac{\partial}{\partial t} + x\tfrac{\partial}{\partial x} + \tfrac{\epsilon}{6}(t^2\tfrac{\partial}{\partial t} - 2tu\tfrac{\partial}{\partial u}), \\ x_4 = x\tfrac{\partial}{\partial x} + u\tfrac{\partial}{\partial u}, \quad x_5 = \epsilon X_1, \quad X_6 = \epsilon X_2, \quad X_7 = \epsilon X_4, \quad X_8 = \epsilon X_3. \end{gathered} \tag{2.24}$$

From Eqs.(2.24) we find

$$X_8 = \epsilon(t\tfrac{\partial}{\partial t} + x\tfrac{\partial}{\partial x}).$$

The followin table of commmutators, evaluated in the first-order of precision, shows that the generators (2.24) span an eight-dimensional approximate Lie algebra, and hence generate an eight-parameter approximate transformations group.

Table 2.1: Approximate Commutators of Approximate symmetry of the perturbed non-linear Wave equation

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ |
|---|---|---|---|---|---|---|---|---|
| $X_1$ | 0 | 0 | $X_1 + \frac{1}{3}(X_8 - X_7)$ | 0 | 0 | 0 | 0 | $X_5$ |
| $X_2$ | 0 | 0 | $X_2$ | $X_2$ | 0 | 0 | $X_6$ | $X_6$ |
| $X_3$ | 0 | 0 | 0 | 0 | $-X_5$ | $-X_6$ | 0 | 0 |
| $X_4$ | 0 | 0 | 0 | 0 | 0 | $-X_6$ | 0 | 0 |
| $X_5$ | 0 | 0 | $X_6$ | $X_6$ | 0 | 0 | 0 | 0 |
| $X_6$ | 0 | 0 | $X_6$ | 0 | 0 | 0 | 0 | 0 |
| $X_7$ | 0 | $-X_6$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_8$ | $-X_5$ | $-X_6$ | 0 | 0 | 0 | 0 | 0 | 0 |

## 2.3 Approximate Symmetries of Gardner Equation by Method I

In this section, we give an example in which the symmetries of the perturbed equation are not inherited from the unperturbed eqution. The authors Khah, Mehdi, and Mokhtary [19] study Gardner equation

$$u_t - 6(u + \epsilon u^2)u_x + u_{xxx} = 0$$

and the unperturbed equation

$$u_t - 6uu_x + u_{xxx} = 0$$

which is well known, as The Kortewege-de Vries(KDV) equation. To see if the symmetries of KDV can be used to generate stable symmetries of Gardner equation.

Let us consider the approximate group generator in the form

$$X = X_0 + \epsilon X_1 = (\xi + \epsilon\alpha)\frac{\partial}{\partial x} + (\tau + \epsilon\beta)\frac{\partial}{\partial t} + (\eta + \epsilon\phi)\frac{\partial}{\partial u} \tag{2.25}$$

Where $\xi, \alpha, \tau, \beta$ and $\phi$ are unknown functions of $x, t$ and $u$. Solving the determining equation

$$X_0{}^{(k)}(F_0(z))|_{\{F_0(z)=0\}} = 0 \tag{2.26}$$

such that

$$F_0(z) + \epsilon F_1(z) = u_t - 6(u + \epsilon u^2)u_x + u_{xxx}$$

The third prolongation of $X_0$.

$$\begin{aligned} X_0{}^3 = \xi\tfrac{\partial}{\partial x} + \tau\tfrac{\partial}{\partial t} + \eta\tfrac{\partial}{\partial u} + \tau_x\tfrac{\partial}{\partial u_x} + \tau_t\tfrac{\partial}{\partial u_t} + \tau_x\tfrac{\partial}{\partial u_x} \\ +\tau_{xx}\tfrac{\partial}{\partial u_{xx}} + \tau_{xt}\tfrac{\partial}{\partial u_{xt}} + \tau_{tt}\tfrac{\partial}{\partial u_{tt}} + \tau_{xxx}\tfrac{\partial}{\partial u_{xxx}} \\ +\tau_{xxt}\tfrac{\partial}{\partial u_{xxt}} + \tau_{xtt}\tfrac{\partial}{\partial u_{xtt}} + \tau_{ttt}\tfrac{\partial}{\partial u_{ttt}} \end{aligned} \tag{2.27}$$

Thus (2.26) gives

$$\begin{aligned} X_0{}^3(F_0(z))|_{F_0(z)=0} = X_0{}^3(u_t - 6uu_x + u_{xxx})|_{u_t=6uu_x-u_{xxx}} = 0, \\ (\eta(-6u_x) + \eta^x(-6u) + \eta^t + \eta^{xxx})|_{\{u_t=6uu_x-u_{xxx}\}} = 0 \end{aligned} \tag{2.28}$$

Substituting $\eta, \eta^x, \eta^t, \eta^{xxx}$ and $u_t = 6uu_x - u_{xxx}$ into (2.28) and comparing the coefficients of the derivatives of $u$ gives

e1: $\tau_u = 0$

e2: $\tau_x = 0$

e3: $\xi_u = 0$

e4: $\eta_{uu} = 0$

e5: $-6\eta - \xi_t + 3\eta_{xxu} - 12u\xi_x - \xi_{xxx} = 0$

e6: $6u\tau_x - \tau_t + 2\xi_x - \tau_{xxx} + \xi_x = 0$

e7: $3\eta_{xu} - 3\xi_{xx} = 0$

e8: $-6u\eta_x + \eta_t + \eta_{xxx} = 0$

By (e2),(e6) reduces to

e9: $\xi_x = \frac{1}{3}\tau_t,$

e10: $\xi_{xx} = 0.$

Using (e8) and (e10) gives

e11: $\eta_{xu} = 0$ implies $\eta = \eta_1(t)u + \eta_2(t).$

Using (e5) and (e11) gives

e12: $-6\eta - \xi_t - 12u\xi_x = 0$

Substituting (e9) into (e12) gives $\quad \eta = -\frac{1}{6}\xi_t - \frac{2}{3}u\tau_t.$

Differentiating (e8) with respect to u and simplifying gives e13: $\tau_{tt} = 0$

e14: $\xi_{xt} = 0$

Using $(e12)_t$ and (e8) gives $\xi_{tt} = 0$.

The determining equations become as following:

$$\begin{aligned}
\xi_u &= 0 \\
\xi_{xx} &= 0 \\
\xi_{tt} &= 0 \\
\xi_x &= \tfrac{1}{3}\tau_t \\
\tau_u &= 0 \\
\tau_x &= 0 \\
\tau_{tt} &= 0 \\
\eta_{uu} &= 0 \\
\eta &= -\tfrac{1}{6}\xi_t - \tfrac{2}{3}u\tau_t.
\end{aligned} \tag{2.29}$$

Hence,

$$\xi = c_1 - 6c_3t + c_4x, \qquad \tau = c_2 + 3c_4t, \qquad \eta = c_3 - 2c_4u, \tag{2.30}$$

where $c_1, c_2, c_3, c_4$ are arbitrary constants.

$$X_0 = (c_1 - 6c_3t + c_4x)\frac{\partial}{\partial x} + (c_2 + 3c_4t)\frac{\partial}{\partial t} + (c_3 - 2c_4u)\frac{\partial}{\partial u}.$$

**2nd Step.** Determination of auxilary function H. For this, we consider

$$H = \frac{1}{\epsilon}[{X_0}^3(u_t - 6uu_x + u_{xxx} - 6\epsilon u^2u_x)]|_{\{u_t = 6uu_x - u_{xxx} - 6\epsilon u^2u_x\}} \tag{2.31}$$

where ${X_0}^3$ is the third prolongation of $X_0$, implies

$$H = \frac{1}{\epsilon}[\eta_0(-6u_x - 12\epsilon uu_x) + {\eta_0}^x(-6u - 6\epsilon u^2) + {\eta_0}^{xx} + {\eta_0}^x]|_{\{u_t=6uu_x-u_{xxx}-6\epsilon u^2u_x\}} \tag{2.32}$$

such that

$$\eta = c_3 - 2c_4u, \quad \eta^x = -3c_4u_x, \quad \eta^{xxx} = -5c_4u_{xxx}, \quad \eta^t = -5c_4u_t + 6c_3u_x.$$

Substituting $\eta, \eta^x, \eta^{xxx}, \eta^t$ and $u_t = 6uu_x - u_{xxx} - 6\epsilon u^2u_x$ into (2.32), hence

$$H = 12uu_x(c_4u - c_3). \tag{2.33}$$

**3rd Step.** Calculate the operators $X_1$ by solving the inhomogeneous determining equation for deformations:

$${X_1}^{(k)}F_0(z)|_{\{F_0(z)=0\}} + H = 0. \tag{2.34}$$

The above determining equation for the unperturbed equation $u_t - 6uu_x + u_{xxx} = 0$ is written as

$${X_1}^3(u_t - 6uu_x + u_{xxx} = 0)|_{\{u_t=6uu_x+u_{xxx})u_x-u_{xxx}\}} + 12uu_x(c_4u - c_3) = 0. \tag{2.35}$$

The left hand side of Eq.(2.35) becomes a polynomail in the variables $u_t, u_x, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{ttt}$. Equating to zero its coefficients we obtain e1: $\beta_u = 0$

e2: $\beta_x = 0$

e3: $\alpha_u = 0$

e4: $\psi_{uu} = 0$

e5: $-6\psi - \alpha_t + 3\psi_{xxu} - 12u\alpha_x - \alpha_{xxx} = 0$

e6: $6u\beta_x - \beta_t + 2\alpha_x - \beta_{xxx} + \alpha_x = 0$

e7: $3\psi_{xu} - 3\alpha_{xx} = 0$

e8: $-6u\psi_x + \psi_t + \psi_{xxx} = 0.$

By (e2),(e6) reduces to

e9: $\alpha_x = \frac{1}{3}\beta_t$

e10: $\alpha_{xx} = 0$

Using (e8) and (e10) gives

e11: $\psi_{xu} = 0$ implies $\psi = \psi_1(t)u + \psi_2(t)$.

Use (e5) and (e11) gives

e12: $-6\psi - \alpha_t - 12u\alpha_x - 12u(c_4u - c_3) = 0$

Substituting (e9) into (e12) gives $\psi = -\frac{1}{6}\alpha_t - \frac{2}{3}u\beta_t$

Differentiate (e8) with respect to u and simplifying gives

e13: $\beta_{tt} = 0,$

e14: $\alpha_{xt} = 0.$

Using $(e12)_t$ and (e8) gives $\alpha_{tt} = 0$. and (e12) yields $c_4 = 0$.

The determining equations become as follows:

$$\begin{aligned}
\alpha_u &= 0 \\
\alpha_{xx} &= 0 \\
\alpha_{tt} &= 0 \\
\alpha_x &= \tfrac{1}{3}\beta_t \\
\beta_u &= 0 \\
\beta_x &= 0 \\
\beta_{tt} &= 0 \\
\psi_{uu} &= 0 \\
\psi &= -\tfrac{1}{6}\alpha_t - \tfrac{2}{3}u\beta_t.
\end{aligned} \tag{2.36}$$

Hence,

$$\alpha = A_1 - 6A_3t + A_4x, \quad \tau = A_2 + 3A_4t, \quad \eta = A_3 - 2(c_3 + A_4)u \tag{2.37}$$

where $A_1, A_2, A_3, A_4$ are arbitrary constants. Hence, we obtain the following approximate symmetries of the Gardner equation

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = (1 - 2\epsilon u)\frac{\partial}{\partial u} - 6\frac{\partial}{\partial t}, \quad X_4 = \epsilon X_1, \quad X_5 = \epsilon X_2,$$

$$X_6 = \epsilon X_3 = \epsilon(\frac{\partial}{\partial u} - 6\frac{\partial}{\partial t}), \quad X_7 = \epsilon(x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u})$$

Because of $c_4 = 0$, the scalling operator

$$X_0{}^4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$$

is not stable, hence, the Gardner equation does not inherit the symmetries of the KdV equation.

The following table of commutators, evaluted in the first-order of precision, shows that the previous generators span an seven-dimensional approximate Lie algebra, and hence generate an seven-parameter approximate transformations group.

Table 2.2: Approximate Commutators of Approximate symmetry of the Gardner equation

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ |
|---|---|---|---|---|---|---|---|
| $X_1$ | 0 | 0 | 0 | 0 | 0 | 0 | $X_4$ |
| $X_2$ | 0 | 0 | $6X_1$ | 0 | 0 | $6X_4$ | $3X_5$ |
| $X_3$ | 0 | $-6X_1$ | 0 | 0 | $-6X_4$ | 0 | $-2X_6$ |
| $X_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_5$ | 0 | 0 | $6X_4$ | $X_6$ | 0 | 0 | 0 |
| $X_6$ | 0 | $-6X_4$ | 0 | 0 | 0 | 0 | 0 |
| $X_7$ | $-X_4$ | $-3X_5$ | $2X_6$ | 0 | 0 | 0 | 0 |

## 2.4 Approximate Symmetries of Potential Burgers Equation by Method I

Potential Burgers equation is [21]

$$F_0(z) + \epsilon F_1(z) = u_t - u_{xx} - \epsilon u_x{}^2 = 0, \tag{2.38}$$

Let us write the approximate group generator in the form

$$X = X^0 + \epsilon X^1 \equiv (\tau_0 + \epsilon\tau_1)\frac{\partial}{\partial t} + (\xi_0 + \epsilon\xi_1)\frac{\partial}{\partial x} + (\eta_0 + \epsilon\eta_1)\frac{\partial}{\partial u}, \tag{2.39}$$

where $\xi_v$, $\tau_v$ and $\eta_v$ $(v = 0, 1)$ are unknown functions of $t, x$, and $u$.

**1st Step.** Solving the following determining equation for the exact symmetries $X^0$ of the unperturbed equation.

$$X^0 F_0(z)|_{F_0(z)=0} = 0 \tag{2.40}$$

The infinitesimal generator for the unperturbed equation is a vector field on a three dimensional space, as there are two independent variables and one dependent variable, therefore,

$$X^0 = \tau_0\frac{\partial}{\partial t} + \xi_0\frac{\partial}{\partial x} + \eta_0\frac{\partial}{\partial u} \tag{2.41}$$

We need to prolong the symmetry generator (2.41) to second order. Thus the prolongation of the infinitesimal symmetry generator as follows

$$X^{0^{[2]}} = X^0 + \eta_o^t\frac{\partial}{\partial u_t} + \eta_o^x\frac{\partial}{\partial u_x} + \eta_o^{tt}\frac{\partial}{\partial u_{tt}} + \eta_o^{xt}\frac{\partial}{\partial u_{xt}} + \eta_o^{xx}\frac{\partial}{\partial u_{xx}}. \tag{2.42}$$

Symmetry criterion (2.40) for the unperturbed partial differential equation gives the relation

$$X^{0^{[2]}}(u_t - u_{xx} - \epsilon {u_x}^2)|_{u_t = u_{xx} + \epsilon {u_x}^2} = 0. \tag{2.43}$$

We obtained the following system of ten determining equations e1: $\eta_{0t} - \eta_{0xx} = 0$

e2: $-\tau_{0t} + \tau_{0xx} + 2\xi_{0x} = 0$

e3: $-\xi_{0t} - 2\eta_{0xu} + \xi_{0xx} = 0$

e4: $-\xi_{0u} + 3\xi_{0u} + 2\tau_{0xu} = 0$

e5: $(-\tau_{0u} + \tau_0 = 0) \equiv 0$

e6: $2\tau_{0u} = 0$

e7: $-\eta_{0uu} + 2\xi_{0xu} = 0$

e8: $\tau_{0uu} = 0$

e9: $\xi_{0uu} = 0$

e10: $2\tau_{0x} = 0.$

From (e6) and (e10) implies

e11: $\tau_0 = A(t)$

Using (e4) and (e11) gives

e12: $\xi_{0u} = 0, \quad \xi_0 = B(t, x)$

From (e7) and (e12), implies

e13: $\eta_{0uu} = 0, \quad \eta_0 = h_1(x, t)u + h_2(x, t)$

Substituting (e11) and (e12) into (e2) gives

e14: $2B_x(x, t) - A_t(t) = 0, \quad (e14)_x \quad \text{gives} \quad 2B_{xx}(x, t) = 0 \quad \text{implies}$

e15: $B(x, t) = B_1(t)x + B_2(t).$

Using (e15) into (e14) implies $2B_1(t) - A_t(t) = 0, \quad thus$

e16: $A = 2\int B_1(t)dt + \alpha_0$.

Using (e11),(e13) and (e15),(e3) reduces to $h_{1t}(t,x) = -\frac{1}{2}B_{1t}(t)x - \frac{1}{2}B_{2t}(t)$,

e17: $h_1(t,x) = -\frac{1}{4}B_{1t}(t)x^2 - \frac{1}{2}B_{2t}(t)x + f(t)$.

Using (e17) in (e13) gives

e18: $\eta_0 = [-\frac{1}{4}B_{1t}(t)x^2 - \frac{1}{2}B_{2t}(t)x + f(t)]u + h_2(t,x)$.

Substituting (e18) into (e1) gives

e19: $-\frac{1}{2}B_{1t}(t) + h_{2xx}(t,x) - [-\frac{1}{4}B_{1tt}(t)x^2 - \frac{1}{2}B_{2tt}(t)x + f_t(t)]u - h_{2t}(t,x) =$

$0, \quad (e19)_u \quad implies$

e20: $-\frac{1}{2}B_{1t}(t) + \frac{1}{4}B_{1tt}(t)x^2 + \frac{1}{2}B_{2tt}(t)x - f_t(t) = 0$

Differentiating (e20) with respect to $x$ twice, gives $B_{1tt}(t) = 0$ which implies

e21: $B_1(t) = \alpha_1 t + \alpha_2$.

Using (e21) in (e20) gives

e22: $-\frac{\alpha}{2} + \frac{1}{2}B_{2tt}(t)x - f_t(t) = 0$.

Differentiating (e22) with respect to $x$ gives $B_{2tt} = 0$

e23: $B_2(t) = \alpha_3 t + \alpha_4$

Subsituting (e23) into (e22) gives

e24: $f(t) = -\frac{\alpha_1 t}{2} + \alpha_5$.

Hence,

$$\begin{aligned}\tau_0 &= \alpha_1 t^2 + 2\alpha_2 t + \alpha_0 \\ \xi_0 &= (\alpha_1 t + \alpha_2)x + \alpha_3 t + \alpha_4 \\ \eta_0 &= (-\tfrac{\alpha_1}{4}x^2 - \tfrac{1}{2}\alpha_3 x - \tfrac{1}{2}\alpha_1 t + \alpha_5)u + h_2(t,x),\end{aligned}$$

$$such \;\; that \;\; h_2(t,x)_{xx} - h_2(t,x)_t = 0. \tag{2.44}$$

Hence,

$$\begin{aligned}X_0 = ((\alpha_1 t + \alpha_2)x + \alpha_3 t + \alpha_4)\tfrac{\partial}{\partial x} + (\alpha_1 t^2 + 2\alpha_2 t + \alpha_0)\tfrac{\partial}{\partial t} \\ +(-\tfrac{\alpha_1}{4}x^2 - \tfrac{1}{2}\alpha_3 x - \tfrac{1}{2}\alpha_1 t + \alpha_5)u + h_2(t,x))\tfrac{\partial}{\partial u}\end{aligned} \tag{2.45}$$

**2nd Step.** Finding the auxilary function H by

$$H = \frac{1}{\epsilon}[{X_0}^{\{2\}}(F_0(z) + \epsilon F_1(z)|_{\{F_o(z)+\epsilon F_1(z)=0\}})] = \frac{1}{\epsilon}[{X_0}^{\{2\}}(u_t - u_{xx} - \epsilon {u_x}^2)|_{u_t = u_{xx} + \epsilon {u_x}^2}] \tag{2.46}$$

Where ${X_0}^{\{2\}}$ is the second prolongation of $X_0$, implies

$$H = \frac{1}{\epsilon}[{\eta_0}^t - 2\epsilon {\eta_0}^x u_x - {\eta_0}^{xx}]|_{\{u_{xx} = u_t - \epsilon {u_x}^2\}}, \tag{2.47}$$

such that

${\eta_0}^x = 2u_x(-\frac{\alpha_1}{4}x^2 - \frac{1}{2}\alpha_3 x - \frac{\alpha_1 t}{2} + \alpha_5) - u_x(\alpha_1 t + \alpha_2) + (-\frac{\alpha_1}{2} - \frac{1}{2}\alpha_3)u + h_{2x}(t,x)$

${\eta_0}^t = 2u_t(-\frac{\alpha_1}{4}x^2 - \frac{1}{2}\alpha_3 x - \frac{\alpha_1 t}{2} + \alpha_5)) - u_t(2\alpha_1 t + 2\alpha_2) - \frac{\alpha_1}{2}u + h_{2t}(t,x) -$

$u_x(\alpha_1 x + \alpha_3)$

${\eta_0}^{xx} = 2u_{xx}(-\frac{\alpha_1}{4}x^2 - \frac{1}{2}\alpha_3 x - \frac{\alpha_1 t}{2} + \alpha_5)) - u_{xx}(2\alpha_1 t + 2\alpha_2) - \frac{\alpha_1}{2}u + h_{2xx}(t,x) -$

$u_x(\alpha_1 x + \alpha_3)$

substituting ${\eta_0}^x, {\eta_0}^t, {\eta_0}^{xx}$ *and* $u_{xx} = u_t - \epsilon {u_x}^2$ into (2.47), we obtain

$$H = (\alpha_1 + \alpha_3)uu_x + (\frac{\alpha_1}{2}x^2 + \alpha_3 x + \alpha_1 t - 2\alpha_5){u_x}^2 - 2h_{2xx}(t,x)u_x. \tag{2.48}$$

**3rd Step.** Calculating the operator $X_1$ by solving the inhomogeneous determining equation for deformations:

$$X^{1\{k\}}(F_0(z))|_{\{F_0(z)=0\}} + H = 0. \tag{2.49}$$

The above determining equation for the current equation is written as

$$\begin{aligned} X^{1\{2\}}(u_t - u_{xx} - \epsilon {u_x}^2)|_{\{u_t = u_{xx} + \epsilon {u_x}^2\}} + (\alpha_1 + \alpha_3)uu_x \\ +(\tfrac{\alpha_1}{2}x^2 + \alpha_3 x + \alpha_1 t - 2\alpha_5){u_x}^2 - 2h_{2xx}(t,x)u_x = 0, \end{aligned} \tag{2.50}$$

where $X^{1\{2\}}$ denotes the second prolongation of the operator

$$X^1 = \tau_1 \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial u} \tag{2.51}$$

Next we compare the coefficients of the derivatives of u in (2.50), we obtain the following ten determining equations

e1: $2{\eta_1}_{ux} - {\xi_1}_{xx} + {\xi_1}_t + (\alpha_1 + \alpha_3)u - 2h_{2x}(t,x) = 0$

e2: $-{\tau_1}_{xx} + {\eta_1}_u - 2{\xi_1}_x - {\eta_1}_u + {\tau_1}_t = 0$

e3: ${\eta_1}_{uu} - 2{\xi_1}_{ux} + (\frac{\alpha_1}{2}x^2 + \alpha_3 x + \alpha_1 t + -2\alpha_5) = 0$

e4: $-2\tau_{1ux} - 3\xi_{1u} + \xi_{1u} = 0$

e5: $-\xi_{1uu} = 0$

e6: $-\tau_{1uu} = 0$

e7: $(-\tau_{1u} + \tau_{1u} = 0) \equiv 0$

e8: $-2\tau_{1u} = 0$

e9: $\eta_{1xx} - \eta_{1t} = 0$

e10: $-2\tau_{1x} = 0$

From (e8) and (e10) implies

e11: $\tau_1 = k(t)$.

Using (e4) and (e11) gives

e12: $\xi_{1u} = 0, \quad \xi_1 = \psi(t, x)$

From (e7) and (e12), implies

e13: $\eta_{1uu} = 0, \quad \eta_1 = g_1(x,t)u + g_2(x,t)$.

Substituting (e11) and (e12) into (e2) gives

e14: $-2\psi_x(t,x) + k_t(t) = 0$

Differentiating (e14) with respect to x, gives $2\psi_{xx} = 0$ implies

e15: $\psi(t,x) = \psi_1(t)x + \psi_2(t)$

Using (e15) in (e14) gives $2\psi_1(t) - k_t(t) = 0$, thus

e16: $k(t) = 2\int \psi_1(t)dt + c_0$

Using (e11),(e13) and (e15),(e1) reduces to $2g_{1x}(t,x) + \psi_{1t}(t)x + \psi_{2t}(t) - 2h_{2x}(t,x) = 0$, which implies e17: $g_1(t,x) = -\frac{1}{4}\psi_{1t}(t)x^2 - \frac{1}{2}\psi_{2t}(t)x - h_2(t,x) + v(t)$.

Using (e17) in (e13) gives

e18: $\eta_1 = [-\frac{1}{4}\psi_{1t}(t)x^2 - \frac{1}{2}\psi_{2t}(t)x - h_2(t,x) + v(t)]u + g_2(t,x)$.

Substituting (e18) into (e9), we obtain

e19: $[-\frac{1}{2}\psi_{1t}(t) - h_{2xx}(x,t)]u + g_{2xx}(t,x) + [\frac{1}{4}\psi_{1tt}(t)x^2 + \frac{1}{2}\psi_{2tt}(t)x + h_{2t}(t,x) - v_t(t)]u - g_{2t}(t,x) = 0$.

Differentiating (e19) with respect to u, we get

e20: $-\frac{1}{2}\psi_{1t}(t) - h_{2xx}(t,x) + \frac{1}{4}\psi_{1tt}(t)x^2 + \frac{1}{2}\psi_{2tt}(t) + h_{2t}(t,x) - v_t(t) = 0$,

since $h_{2xx}(t,x) - h_{2t}(t,x) = 0, \quad implies$

e21: $\frac{1}{2}\psi_{1t}(t) + \frac{1}{4}\psi_{1tt}(t)x^2 + \frac{1}{2}\psi_{2tt}(t)x + h_{2t}(t,x) - v_t(t) = 0$

Differentiate (e21) with respect to $x$ twice, gives $\psi_{1tt}(t) = 0$, implies

e22: $\psi_1(t) = c_1t + c_2$

Using (e22) in (e21) gives

e23: $\frac{c_1}{2} - \frac{1}{2}\psi_{2tt}(t)x + v_t(t) = 0$.

Differentiating (e23) with respect to $x$ , gives $\psi_{2tt} = 0$, implies

e24: $\psi_2(t) = c_3t + c_4$.

Using (e24) in (e23) gives $\frac{c_1}{2} + v_t(t) = 0$, implies

e25: $v(t) = -\frac{c_1t}{2} + c_5$

Substituting all these into (e21) for consistency check, we deduce

$[\frac{1}{2}(c_1) - \frac{1}{4}(0)x^2 - \frac{1}{2}(0)x - \frac{c}{2}] \equiv 0$.

Now, we address (e20), to set: e26: $g_{2t}(t,x) - g_{2xx}(t,x) = 0$.

Hence,

$$
\begin{aligned}
\tau_1 &= 2(\tfrac{c_1}{2}t^2 + c_2 t) + c_0 \\
\xi_1 &= (c_1 t + c_2)x + (c_3 t + c_4) \\
\eta_1 &= [-\tfrac{1}{4}c_1 x^2 - \tfrac{1}{2}c_3 x - \tfrac{c_1}{2}t + c_5 - h - 2(t,x)]u + g_2(t,x) \\
such \;\; that & \\
g_{2t}(t,x) - g_{2xx}(t,x) &= 0, \\
h_{2t}(t,x) - h_{2xx}(t,x) &= 0.
\end{aligned} \tag{2.52}
$$

## 2.5 Approximate Symmetries of Potential Burgers Equation using an Alternative Method

In this section, we use **Method III**, which is used in [21] for same problem. We are expanding the dependent variable in a perturbation expansion,

$$u = u_0 + \epsilon u_1 + \cdots .$$

Substituting this expansion into the original equation, we obtain

$$u_{0t}(t,x)+\epsilon u_{1t}(t,x)-u_{0xx}(t,x)-\epsilon u_{1xx}(t,x)-\epsilon(u_{0x}(t,x)^2+2\epsilon u_{0x}(t,x)u_{1x}(t,x)+\epsilon^2 u_{1x}(t,x)^2) = 0.$$

Since, we are considering first-order of precision, the above equation becomes

$$u_{0t}(t,x) + \epsilon u_{1t}(t,x) - u_{0xx}(t,x) - \epsilon u_{1xx}(t,x) - \epsilon u_{0x}(t,x)^2 = 0.$$

This gives,

$$\begin{aligned} Order \;\; 1 \;\; &: \qquad u_{0t}(t,x) - u_{0xx}(t,x) = 0, \\ Order \;\; \epsilon \;\; &: \qquad u_{1t}(t,x) - u_{1xx}(t,x) = (u_{0x}(t,x)^2). \end{aligned} \tag{2.53}$$

So that at order 1, $h(t,x) = 0$ and at order $\epsilon$, $h(t,x) = (u_{0x}(t,x))^2$, a known function to be obtain from the first equation. Now, we need to calculate the symmetries of the following equation

$$u_t(t,x) - u_{xx}(t,x) = h(t,x). \tag{2.54}$$

The corresponding generator is

$$X = \tau\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u},$$

where $\tau, \xi$ and $\eta$ are unknown functions of $t, x$ and $u$, Then, expanding the generator $X$ into the second order

$$X^2 = \tau\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}},$$

and applying to (2.54)

$$X^2(u_t(t,x) - u_{xx}(t,x) - h(t,x))|_{\{u_t(t,x)=u_{xx}(t,x)-h(t,x)\}} = 0 \tag{2.55}$$

The above equation reduces to

$$h_x(t,x)\xi + h_t(t,x) + \eta^t - \eta^{xx} = 0. \tag{2.56}$$

Now, substituting $\eta^x$ and $\eta^{xx}$ into (2.56) and comparing the coefficients of the derivatives of $u$ we get

e1: $-\xi_t - h(t,x)\xi_u - 2\eta_{xu} + 2h(t,x)\tau_{xu} + \xi_{xx} = 0$

e2: $\eta_u - \tau_t - 2h(t,x)\tau_u - \eta_u + h(t,x)\tau_u + 2\xi_x + \tau_{xx} = 0$

e3: $-\eta_{uu} + h(t,x)\tau_{uu} + 2\xi_{xu} = 0$

e4: $-\xi_u + 3\xi_u + 2\tau_{xu} = 0$

e5: $\xi_{uu} = 0$

e6: $\tau_{uu} = 0$

e7: $2\tau_x = 0$

e8: $(-\tau_u + \tau_u) \equiv 0$

e9: $2\tau_u = 0$

e10: $h_x(t,x)\xi + h_t(t,x)\tau + \eta_t + h(t,x)\eta_u - h(t,x)\eta_t - h^2(t,x)\tau_u - \eta_{xx} + h(t,x)\eta_{xx} = 0$

From (e7) and (e9) implies

e11: $\tau = A(t)$.

Using (e4) and (e11) gives $\xi_u = 0$, which implies

e12: $\xi = B(t,x)$

From (e3) and (e12), we get

e13: $\eta_{uu} = 0, \quad \eta = \phi_1(t,x)u + \phi_2(t,x)$.

Substituting (e11) and (e12) into (e2) gives

e14: $-2B(t,x) + A_t(t) = 0$.

Differentiating both sides of (e14) with repect to $x$ gives $2B_{xx}(t,x) = 0$, which implies

e15: $B(t,x) = B_1(t)x + B_2(t)$

Using (e15) in (e14) gives $2B_1(t) - A_t(t) = 0$, and so

e16: $A = 2\int B_1(t)dt + a_0$.

Using (e11),(e13) and (e15),(e1) reduces to $2\phi_{1x}(t,x) = -\frac{1}{2}B_{1t}(t)x - \frac{1}{2}B_{2t}(t)$, which implies,

e17: $\phi_1(t,x) = -\frac{1}{4}B_{1t}(t)x^2 - \frac{1}{2}B_{2t}(t)x + f(t)$.

Using (e17) in (e13) gives

e18: $\eta = [-\frac{1}{4}B_{1t}(t)x^2 - \frac{1}{2}B_{2t}(t)x + f(t)]u + h_2(t,x)$.

Substituting (e18) into (e10), gives

e19: $h_x(t,x)\xi + h_t(t,x)\tau + [-\frac{1}{4}B_{1tt}(t)x^2 - \frac{1}{2}B_{2tt}(t)x + f_t(t)]u + \phi_{2t}(t,x) + h(t,x)[-\frac{1}{4}B_{1t}(t)x^2 - \frac{1}{2}B_{2t}(t)x + f(t)] - h(t,x)[2B_1(t)] + \frac{1}{2}B_{1t}(t)u - \phi_{2xx}(t,x) + h(t,x)\tau_{xx} = 0$

Differentiating (e19) with respect to u, we obtain

e20: $-\frac{1}{4}B_{1tt}(t)x^2 - \frac{1}{2}B_{2tt}(t)x + f_t(t) + \frac{1}{2}B_{1t}(t) = 0.$

Differentiating (e19) with respect to $x$ twice, gives $B_{1tt}(t) = 0$, which implies

e21: $B_1(t) = a_1 t + a_2$

Using (e21) in (e20), gives

e22: $\frac{1}{2}a_1 - \frac{1}{2}B_{2tt}(t)x + f_t(t) = 0$

Differentiate (e22) with respect $x$, gives $B_{2tt}(t) = 0$, and so,

e23: $B_2(t) = a_3 t + a_4.$

using (e23) back in (e22), we obtain

e24: $f(t) = -\frac{a_1}{2}t + a_5.$

Substituting all these into (e20) for consistency check,

$$[-\frac{1}{2}a_1 + \frac{1}{4}(0)x^2 - \frac{1}{2}(0)x + \frac{a_1}{2}] \equiv 0.$$

Now, we address (e19) to set

$h_x(t,x)[a_1 tx + a_2 x + a_3 t + a_4] + h_t(t,x)[a_1 t^2 + 2a_2 t + a_0] - \frac{a_1}{2}u + \phi_{2t}(t,x) + h(t,x)[-\frac{1}{4}a_1 x^2 - \frac{1}{2}a_3 x - \frac{a_1}{2}t + a_5] - h(t,x)[2a_1 t + 2a_2] + \frac{1}{2}[a_1 t + a_2]u - \phi_{2xx}(t,x) = 0$

Finally, we obtain

$$\begin{aligned} \xi &= a_1 xt + a_2 x + a_3 t + a_4 \\ \tau &= a_1 t^2 + 2a_2 t + a_0 \\ \eta &= [-\tfrac{1}{4}a_1 x^2 - \tfrac{1}{2}a_3 x - \tfrac{a_1}{2}t + a_5]u + \phi_2(t,x) \end{aligned} \tag{2.57}$$

Such that

$$\begin{aligned} \phi_{2t}(t,x) - \phi_{2xx}(t,x) - h(t,x)[\tfrac{1}{4}a_1x^2 + \tfrac{1}{2}a_3x + \tfrac{5}{2}a_1t + 2a_2 - a_5 \\ = h_x(t,x)[a_1tx + a_2x + a_3t + a_4] + h_t(t,x)[a_1t^2 + 2a_2t + a_0] + \tfrac{1}{2}a_2u \end{aligned} \tag{2.58}$$

Where $a_0, a_1, a_2, a_3, a_4$ and $a_5$ are arbitrary constants.

## 2.6 Approximate Solution of Perturbed Non-linear Wave Equation

Consider Eq.(2.9),

$$u_{tt} - (u^2u_x)_x + \epsilon u_t = 0. \tag{2.59}$$

Using the symmetry $X = X_3 - X_4$ with $X_3, X_4$ from (2.24),

$$X = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + \frac{\epsilon}{6}(t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}) \tag{2.60}$$

The approximate invariants of (2.60) can written as

$$E(t,x,u,\epsilon) = E^0(t,x,u) + \epsilon E^1(t,x,u) + O(\epsilon).$$

which in term of Eq.(2.60) is

$$X = X^0 + \epsilon X^1,$$

such that

$$X^0 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \quad X^1 = \frac{1}{6}(t^2\frac{\partial}{\partial t} - 2tu\frac{\partial}{\partial u}).$$

As seen ealier, these lead to the system

$$\begin{aligned} t\tfrac{\partial E^0}{\partial t} - u\tfrac{\partial E^0}{\partial u} &= 0, \\ t\tfrac{\partial E^1}{\partial t} - u\tfrac{\partial E^1}{\partial u} &= -\tfrac{1}{6}(t^2\tfrac{\partial E^0}{\partial t} - 2tu\tfrac{\partial E^0}{\partial u}). \end{aligned} \tag{2.61}$$

Solving Eqs.(2.61), we will have two functionally independent invariants

$$\begin{aligned} E_1 &= {E_1}^0(t,x,u) + \epsilon {E_1}^1(t,x,u) \\ E_2 &= {E_2}^0(t,x,u) + \epsilon {E_2}^1(t,x,u) \end{aligned} \tag{2.62}$$

for generator (2.60).

**Remark 2** *Functions* (2.62) *are said to be functionally dependent if* $E_2 = \psi(E_1)$*, equivalently we can say*

$${E_2}^0(t,x,u) + \epsilon {E_2}^1(t,x,u) = \psi({E_1}^0(t,x,u) + \epsilon {E_1}^1(t,x,u)) + o(\epsilon)$$

*if such an* $\psi$ *does not exist, then* $E_1$ *and* $E_2$ *are functionally independent.*

The first equation in (2.61) has two functionally independent solutions,

$${E_1}^0 = x, \quad {E_2}^0 = tu$$

Substituting ${E_1}^0 = x$ into the second equation in (2.61) and taking its simplest solution ${E_1}^1 = 0$, we obtain one invariant in (2.62),

$$E_1 = x. \tag{2.63}$$

Note that it does not involve the dependent variable $u$. Now we substitute the solution ${E_2}^0 = tu$ of the first equation in (2.61) into the second equation in (2.61) and get non-homogeneous linear equation:

$$t\frac{\partial {E_1}^1}{\partial t} - u\frac{\partial {E_2}^1}{\partial u} = \frac{1}{6}(t^2 u).$$

The corresponding characteristic equation are

$$\frac{dt}{t} = -\frac{du}{u} = 6\frac{d{E_2}^1}{t^2 u},$$

which the first integral $tu = \lambda = const$. Therefore, the second equation

$$\frac{dt}{t} = 6\frac{d{E_2}^1}{t^2 u}$$

gives

$${E_2}^1 = \frac{1}{6}t^2 u + c. \tag{2.64}$$

Assuming that $c = 0$, we get the second invariant in (2.62),

$$E_2 = tu + \frac{\epsilon}{6}t^2 u.$$

By remark 2, invariants (2.63), (2.64) are functionally independent. Letting $E_2 = \phi(E_1)$, *i.e.*,

$$(1+\frac{\epsilon}{6}t)tu = \phi(x)$$

and solving for $tu$ in the first order of precision,

$$tu = (1+\frac{\epsilon t}{6})^{-1}\phi(x) = (1-\frac{\epsilon t}{6})\phi(x) + o(\epsilon),$$

We get the approximately invariant solution:

$$u = (\frac{1}{t} - \frac{\epsilon}{6})\phi(x). \tag{2.65}$$

Differentiation of (2.65) gives:

$$u_t = -\frac{1}{t^2}\phi, \quad u_{tt} = \frac{2}{t^3}\phi, \quad u_x = (\frac{1}{t} - \frac{\epsilon}{6})\phi'. \tag{2.66}$$

where prime denotes derivatives with respect to $x$. Moreover,

$$u^2u_x = (\frac{1}{t} - \frac{\epsilon}{6})^3\phi^2\phi' = (\frac{1}{t^3} - \frac{\epsilon}{2t^2})\phi^2\phi' + o(\epsilon)$$

and therefore we have in our approximation:

$$(u^2u_x)_x = (\frac{1}{t^3} - \frac{\epsilon}{2t^2})(\phi^2\phi')'. \tag{2.67}$$

Substituting these into Eq. (2.9), we get

$$(\phi^2\phi^{'})^{'} = 2\phi. \tag{2.68}$$

By integrating Eq.(2.68), reduced to a first-order equation by substitution $\phi^{'} = p(\phi)$ and denoting $p^2 = v$ , respectively, we obtain

$$\phi\frac{dv}{d\phi} + 4v = 4$$

Then integrating the above resulting linear equation and substituting back $v, p$ as a function of $\phi$, we obtain

$$\frac{d\phi}{dx} = \pm\sqrt{1 + c\phi^{-4}}.$$

The integration provides the general solution to Eq.(2.68), setting $c = 0$, we have $\phi(x) = \pm x$, and the invariant solution (2.65) is

$$u = \pm(\frac{x}{t} - \epsilon\frac{x}{6}).$$

## 2.7 Approximate Solution of Potential Burgers Equation

We now turn our attention to the Approximate Solution of potential Burgers equation (2.38) which is given in [21]. Solution corresponding to exact symmetry will be derived from the approximate symmetries given in Equation (2.44), selecting parameter $\alpha_2$ and all others to be zero. The characteristic equation are

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{0} \tag{2.69}$$

from which the similarity variables are

$$\xi = \frac{x}{\sqrt{t}}, \qquad u = f(\xi). \tag{2.70}$$

Substituting into the original equation, one has

$$f'' + \frac{1}{2}\xi f' + \epsilon(f')^2 = 0, \tag{2.71}$$

where prime denotes derivatives with respect to $\xi$. The solution is

$$f = \int_0^{\xi} \frac{exp(-\bar{\xi}^2/4)}{\epsilon \int_0^{\bar{\xi}} exp(-\eta^2/4)d\eta + {c_1}^{-1}} d\bar{\xi} + c_2, \tag{2.72}$$

where $c_1$ and $c_2$ are constants. This solution is an exact solution of the original problem. To obtain an approximate solution up to first-order, one may expand the denominator and keep terms up to order $\epsilon$

$$u = f = c_1 \int_0^{\xi} exp(-\bar{\xi}^2/4)d\bar{\xi} + c_2 - \epsilon {c_1}^2 \int_0^{\xi} \left[ exp(-\bar{\xi}^2/4) \int_0^{\bar{\xi}} exp(-\eta^2/4)d\eta \right] d\bar{\xi}, \tag{2.73}$$

For Method I, the approximate symmetries are given in Eqs. (2.52), (2.44). Selecting $\alpha_2$ and $c_2$ parameters and all other to be zero, one has

$$\frac{dx}{(\alpha_2 + \epsilon c_2)x} = \frac{dt}{2t(\alpha_2 + \epsilon c_2)} = \frac{du}{0}. \tag{2.74}$$

Note that selecting $c_2 = 0$ or $c_2 \neq 0$ does not change the transformations. This is why we call such symmetries denoted by capital letters as trivial symmetries. Equation (2.74) yields the same similarity variables as in the exact symmetry case. Hence the solution(2.72) and this solution is not a proper first-order approximation and contains term of o($\epsilon^2$) and Higher. For Method III, the equations to be considered are qiven in (2.53). The first-order solution is obtained by using $a_2$ parameter of symmetries given in Equation (2.57) with $h = 0$ in Equation (2.58)

$$u_0 = c_1 \int_0^{\xi} exp(-\bar{\xi}^2/4)d\bar{\xi} + c_2. \tag{2.75}$$

such that

$$h = \left(\frac{\partial u_0}{\partial x}\right)^2 = \frac{{c_1}^2}{t} exp(-\xi^2/2). \tag{2.76}$$

Taking all parameter except $a_2$ zero in Equation (2.58), one can verify that the specific $h$ defined above satisfies Equation (2.58). Selecting parameter $a_2$ in the system given in (2.57), one has

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du_1}{0} \tag{2.77}$$

from which the similarity variables are

$$\xi = \frac{x}{\sqrt{t}}, \qquad u_1 = g(\xi). \tag{2.78}$$

Substituting Equation (2.78) and (2.76) into Equation (2.53) and solving one has

$$u_1 = g = -{c_1}^2 \int_0^{\xi} \left[ exp(-\bar{\xi}^2/4) \int_0^{\bar{\xi}} exp(-\eta^2/4) d\eta \right] d\bar{\xi} + c_3 \int_0^{\xi} exp(-\bar{\xi}^2/4) d\bar{\xi} + c_4. \tag{2.79}$$

Equations (2.75) and (2.79) are the first and second terms in the perturbation expansions. Redefining the constatnts, the approximate solution written as follows

$$u = c_1 \int_0^{\xi} exp(-\bar{\xi}^2/4) d\bar{\xi} + c_2 - \epsilon {c_1}^2 \int_0^{\xi} \left[ exp(-\bar{\xi}^2/4) \int_0^{\bar{\xi}} exp(-\eta^2/4) d\eta \right] d\bar{\xi}. \tag{2.80}$$

## CHAPTER 3

# PERTURBED NON-LINEAR (2+1) DIMENSIONAL WAVE EQUATION

## 3.1 Introduction

In this chapter we study the approximate symmetries of perturbed non-linear (2+1) dimensional wave equation using one of the approximate Lie symmetry Methods. Moreover approximate invariant solutions of the perturbed non-linear wave equation based on the Lie group method are constructed.

One may refer the reader for some cases of studying unperturbed and perturbed non-linear wave equations. Bokhari, Kara, Karim, Zaman [7] discussed symmetries of unperturbed nonlinear wave equation that arise as consequence of some Riemannian metrics of signature-2. Zhi-Yong, Yu-Fu and Xue-Lin [26] gave approximate symmetries classification of perturbed nonlinear one dimensional wave equation with approximate invariants solutions of the equations. Later on, Ahmed, Bokhari, Kara and Zaman [1] gave a classification of symmetries of unperturbed nonlinear (2+1) dimensional wave equation with respective commutator tables. Zhang, Gao and Chen [27] discussed approximate symmetry classification of perturbed nonlinear one dimensional wave equation with arbitrary function performed by means of the method originated from Fushchich and Shtelen [12].

The perturbed nonlinear wave equation:

$$u_{tt} + \epsilon u_t = (f(u)u_x)_x + (g(u)u_y)_y \tag{3.1}$$

For arbitrary $f(u) = g(u) = u$ the Eq.(3.1) becomes as follows:

$$u_{tt} + \epsilon u_t = (uu_x)_x + (uu_y)_y \tag{3.2}$$

In this case, we use Method I to obtain complete approximate symmetry classification of Eq.(3.2) with the first order of precision $o(\epsilon)$.

Let us consider the approximate group generators in the form

$$X = X_0 + \epsilon X_1 = (\tau_0 + \epsilon\tau_1)\frac{\partial}{\partial t} + (\xi_0 + \epsilon\xi_1)\frac{\partial}{\partial x} + (\theta_0 + \epsilon\theta_1)\frac{\partial}{\partial y} + (\eta_0 + \epsilon\eta_1)\frac{\partial}{\partial u}, \tag{3.3}$$

where $\tau_i, \xi_i, \theta_i$ and $\eta_i$ for $i = 0, 1.$ are unknown functions of $t, x, y$ and $u$.

## 3.2 Exact Symmetries

To find the exact symmetries we need to solve the determining equation

$$X_0{}^{(2)} F_0(z)|_{F_0(z)=0} = 0, \tag{3.4}$$

where $F_0(z) = u_{tt} - (uu_x)_x - (uu_y)_y$, which the unperturbed part of the Eq.(3.2) and $X_0{}^{(2)}$ is the second prolongation of the infinitesimal generator $X_0$, given as follow

$$\begin{aligned} X_0{}^2 = X_0 + \eta_0{}^t \tfrac{\partial}{\partial u_t} + \eta_0{}^x \tfrac{\partial}{\partial u_x} + \eta_0{}^y \tfrac{\partial}{\partial u_y} \\ + \eta_0{}^{tt} \tfrac{\partial}{\partial u_{tt}} + \eta_0{}^{tx} \tfrac{\partial}{\partial u_{tx}} + \eta_0{}^{ty} \tfrac{\partial}{\partial u_{ty}} \\ + \eta_0{}^{xx} \tfrac{\partial}{\partial u_{xx}} + \eta_0{}^{ty} \tfrac{\partial}{\partial u_{ty}} + \eta_0{}^{yy} \tfrac{\partial}{\partial u_{yy}}. \end{aligned} \tag{3.5}$$

This implies that the determining Eq.(3.4), take the form

$$\left(\eta_0(-u_{xx} - u_{yy}) + \eta_0{}^x(-2u_x) + \eta_0{}^y(-2u_y) + \eta_0{}^{xx}(-u) + \eta_0{}^{yy}(-u) + \eta_0{}^{tt}\right)|_{u_{tt}=(uu_x)_x+(uu_y)_y} = 0, \tag{3.6}$$

where the functions $\eta_0{}^x, \eta_0{}^y, \eta_0{}^t, \eta_0{}^{xx}, \eta_0{}^{yy}$ and $\eta_0{}^{tt}$ are given by

$$\begin{aligned} \eta_0{}^x &= D_x\eta_0 - (u_x D_x\xi_0 + u_y D_x\theta_0 + u_t D_x\tau_0), \\ \eta_0{}^y &= D_y\eta_0 - (u_y D_y\xi_0 + u_y D_y\theta_0 + u_t D_y\tau_0), \\ \eta_0{}^t &= D_t\eta_0 - (u_x D_t\xi_0 + u_y D_t\theta_0 + u_t D_t\tau_0), \\ \eta_0{}^{xx} &= D_x\eta_0{}^x - u_{xx} D_x\xi_0 - u_{xy} D_x\theta_0 - u_{xt} D_x\tau_0, \\ \eta_0{}^{yy} &= D_y\eta_0{}^y - u_{xy} D_y\xi_0 - u_{yy} D_y\theta_0 - u_{yt} D_y\tau_0, \\ \eta_0{}^{tt} &= D_t\eta_0{}^t - u_{xt} D_t\xi_0 - u_{yt} D_t\theta_0 - u_{tt} D_t\tau_0, \end{aligned} \tag{3.7}$$

here $D_x, D_y$ and $D_t$ denote the total derivative operators with respect to $x, y$ and $t$, respectively and take the form

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + u_{xt} \frac{\partial}{\partial u_t} + \cdots + u_{xtt} \frac{\partial}{\partial u_{tt}}, \\ D_y &= \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + u_{ty} \frac{\partial}{\partial u_t} + \cdots + u_{ytt} \frac{\partial}{\partial u_{tt}}, \\ D_t &= \frac{\partial}{\partial y} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{yt} \frac{\partial}{\partial u_y} + u_{tt} \frac{\partial}{\partial u_t} + \cdots + u_{ttt} \frac{\partial}{\partial u_{tt}}. \end{aligned} \tag{3.8}$$

Now, substituting ${\eta_0}^x, {\eta_0}^y, {\eta_0}^t, {\eta_0}^{xx}, {\eta_0}^{yy}, {\eta_0}^{xt}, {\eta_0}^{ty}$ and ${\eta_0}^{xy}$ into Eq.(3.6) and comparing the coefficients of the derivatives of $u$ gives the following system of determining equations

e1: $\xi_{ou} = 0$

e2: $\xi_{0t} = 0$

e3: $\tau_{0u} = 0$

e4: $\tau_{0y} = 0$

e5: $\tau_{0x} = 0$

e6: $\theta_{0u} = 0$

e7: $\theta_{0t} = 0$

e8: $\eta_{0uu} = 0$

e9: $-2\eta_{0x} - 2u\eta_{0xu} + u\xi_{0xx} + u\xi_{0yy} = 0$

e10: $-\eta_{0u} + 2\xi_{0x} - 2\tau_{0t} = 0$

e11: $-2\eta_{0y} + u\theta_{0xx} + u\theta_{0yy} = 0$

e12: $-2\eta_{0y} + 2\theta_{0y} + \eta_{0u} - 2\tau_{0t} = 0$

e13: $-u\eta_{0xx} + \eta_{0tt} - u\eta_{0yy} = 0$

e14: $2\eta_{0ut} - \tau_{0tt} = 0$

e15: $2u\xi_{0x} - 2u\tau_{0t} - \eta_0 = 0$

e16: $\theta_{0x} + \xi_{0y} = 0$

e17: $-2u\tau_{0t} + 2u\theta_{0y} - \eta_0 = 0$

Form (e1) and (e2), we obtain

e18: $\xi_0 = F(x, y)$.

From (e3),(e4) and (e5), we get

e19: $\tau_0 = A(t)$.

Using (e5) and (e6), to get

e21: $\theta_0 = B(x, y)$.

Differentiating (e12) with respect to $u$, gives $-\eta_{0uy} = 0$, which implies

e21: $\eta_0 = \phi_1(t, x)u + \phi_2(t, x)$.

Differentiating (e17) with respect to $u, y$ respectively, implies $\theta_{0yy} = 0$.

Differentiating (e10) with respect to $y$, gives $\xi_{0xy} = 0$.

Similarly, differentiating (e11) with respect to $x$, we deduce $\theta_{0xx} = 0$.

From (e15) and (e10), subtracting one from the other , to get

e22: $\xi_{0x} - \theta_{0y} = 0$

Differentiating (e16) with respect to $y$ and differentiating (e22) with respect to $x$, then subtracting the resulting equations one from the other, we obtain

e23: $\xi_{0xx} + \xi_{0yy} = 0$.

Using (e9) and (e23), we get $\eta_{0xu} = 0$.

Differentiating (e15) with respect to $u$ and $x$ respectively, implies $\xi_{0xx} = 0$

Differentiating (e17) with respect to $u$ and $x$ respectively , to get

e24: $\theta_{0xy} = 0$

From (e16) and (e24), we get $\xi_{0yy} = 0$.

From (e14), imlpies $\tau_{0tt} = 2\eta_{0ut}$, substituting into (e17), then differentiating the resulting equation with respect to $u, t$ respectively , we deduce $\eta_{0tu} = 0$ and $\tau_{0tt} = 0$.

Finally, from all the previous, we obtain

$$\begin{aligned} \xi_0 &= a_3 x + a_1 y + a_2 \\ \theta_0 &= a_3 y - a_1 x + a_6 \\ \tau_0 &= a_4 t + a_5 \\ \eta_0 &= 2u(a_3 - a_4) \end{aligned} \tag{3.9}$$

where $a_1, a_2, a_3, a_4, a_5$and $a_6$ are arbitrary constants, hence

$$X_0 = (a_4 t + a_5)\frac{\partial}{\partial t} + (a_3 x + a_1 y + a_2)\frac{\partial}{\partial x} + (a_3 y - a_1 x + a_6)\frac{\partial}{\partial y} + (2u(a_3 - a_4))\frac{\partial}{\partial u} \tag{3.10}$$

## 3.3 Approximate Symmetries

First we need to determine the auxiliary function $H$ by the equation

$$H = \frac{1}{\epsilon}\left[{X_0}^{(k)}(F_0(z) + \epsilon F_1(z))|_{F_0(z)+\epsilon F_1(z)=0}\right].$$

Substituting the formula (3.10) of the generator $X_0$ and $F_0(z) + \epsilon F_1(z) = u_{tt} + \epsilon u_t - (uu_x)_x - (uu_y)_y$ into above equation we obtain the auxiliary function

$$H = a_4 u_t. \tag{3.11}$$

Now, calculating the operator $X_1$ by solving the inhomogeneous determining equation for deformations:

$${X_1}^{(k)} F_0(z)|_{F_0(z)} + H = 0$$

The determining equation for the above equation written as

$$\left[{X_1}^{(2)}(u_{tt} - (uu_x)_x - (uu_y)_y)|_{u_{tt}=(uu_x)_x+(uu_y)_y}\right] + a_4 u_t = 0 \tag{3.12}$$

The left hand side of Eq.(3.12) becomes a ploynomail in the variables $u_t, u_x, u_y, u_{xx}, u_{yy}, u_{tt}, u_{xy}, u_{xt}, u_{ty}$. Equating the coefficients to zero we obtain the following system of determining equations

e1: ${\xi_1}_u = 0$

e2: ${\xi_1}_t = 0$

e3: $\tau_{1u} = 0$

e4: $\tau_{1y} = 0$

e5: $\tau_{1x} = 0$

e6: $\theta_{1u} = 0$

e7: $\theta_{1t} = 0$

e8: $\eta_{1uu} = 0$

e9: $-2\eta_{1x} - 2u\eta_{1xu} + u\xi_{1xx} + u\xi_{1yy} = 0$

e10: $-\eta_{1u} + 2\xi_{1x} - 2\tau_{1t} = 0$

e11: $-2\eta_{1y} + u\theta_{1xx} + u\theta_{1yy} = 0$

e12: $-2\eta_{1y} + 2\theta_{1y} + \eta_{1u} - 2\tau_{1t} = 0$

e13: $-u\eta_{1xx} + \eta_{1tt} - u\eta_{1yy} = 0$

e14: $2\eta_{1ut} - \tau_{1tt} + a_4 = 0$

e15: $2u\xi_{1x} - 2u\tau_{1t} - \eta_1 = 0$

e16: $\theta_{1x} + \xi_{1y} = 0$

e17: $-2u\tau_{1t} + 2u\theta_{1y} - \eta_1 = 0$

Form (e1) and (e2), implies

e18: $\xi_1 = G(x, y)$

From (e3),(e4) and (e5), we obtain

e19: $\tau_1 = K(t)$

Using (e5) and (e6), we get

e21: $\theta_1 = L(x, y)$

Differentiating (e12) with respect to $u$, gives $-\eta_{0uy} = 0$, which implies

e21: $\eta_1 = \varphi_1(t, x)u + \varphi_2(t, x)$

Differentiating (e17) with respect to $u$ and $y$ respectively, implies $\theta_{1yy} = 0$

Differentiating (e10) with respect to $y$, gives $\xi_{1xy} = 0$

Similarly, differentiating (e11) with respect to $x$, we deduce $\theta_{1xx} = 0$.

From (e15) and (e10), subtracting one from the other , to get

e22: $\xi_{1_x} - \theta_{1_y} = 0$

Differentiating (e16) with respect to $y$ and Differentiating (e22) with respect to $x$, then subtract the resulting equations one from the other, we find

e23: $\xi_{1xx} + \xi_{1yy} = 0$

Using (e9) and (e23), we obtain $\eta_{1xu} = 0$

Differentiating (e15) with respect to $u, x$ respectively, implies $\xi_{1xx} = 0$

Differentiating (e17) with respect to $u, x$ respectively , we obtain

e24: $\theta_{1xy} = 0$

From (e16) and (e24), we get $\xi_{1yy} = 0$, differentiating (e15) with respect to $u, t$ respectively, implies $\tau_{1tt} = 2\eta_{1ut}$ , substituteing into (e14), to get $\tau_{1tt} = \frac{a_4}{5}$, which implies

e25: $\tau_1 = \frac{a_4}{10}t^2 + b_4 t + b_5$

Finally, from all the previous, we obtain

$$
\begin{aligned}
\xi_1 &= b_3x + b_1y + b_2 \\
\theta_1 &= b_3y - b_1x + b_6 \\
\tau_1 &= \tfrac{1}{10}a_4t^2 + b_4t + b_5 \\
\eta_1 &= 2u(b_3 - \tfrac{1}{5}a_4t - b_4)
\end{aligned}
\tag{3.13}
$$

where $b_1, b_2, b_3, b_4, b_5$and $b_6$ are arbitrary constants. Hence we obtain the following approximate symmetries of the perturbed nonlinear (2-1) dimensional wave equation:

$$
X_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}
$$

$$
X_4 = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} + \epsilon(\frac{t^2}{10}\frac{\partial}{\partial t} - \frac{2}{5}tu\frac{\partial}{\partial u}), \quad X_5 = \frac{\partial}{\partial t}, \quad X_6 = \frac{\partial}{\partial y}, \quad X_7 = \epsilon X_1
$$

$$
X_8 = \epsilon X_2, \quad X_9 = \epsilon X_3, \quad X_{10} = \epsilon X_5, \quad X_{11} = \epsilon X_6, \quad , X_{12} = \epsilon X_4.
$$

**Remark 3**

$$
X_{12} = \epsilon \left( t\frac{\partial}{\partial t} - 2ut\frac{\partial}{\partial u} \right)
$$

The following table of commmutators, computed in the first-order of precision, shows that the previous generators span a twelve-dimensional approximate Lie algebra, and hence generate a twelve-parameter approximate transformations group.

Table 3.1: Approximate Commutators Table of Approximate Symmetries of the Perturbed (2+1) Dimensional Non-linear Wave Equation

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_1$ | 0 | $X_6$ | 0 | 0 | 0 | $-X_2$ | 0 | $X_{11}$ | 0 | 0 | 0 | 0 |
| $X_2$ | $-X_6$ | 0 | $X_2$ | 0 | 0 | 0 | $-X_{11}$ | 0 | $X_8$ | 0 | 0 | 0 |
| $X_3$ | 0 | $-X_2$ | 0 | 0 | 0 | $-X_6$ | 0 | $-X_8$ | 0 | 0 | $X_{11}$ | 0 |
| $X_4$ | 0 | 0 | 0 | 0 | $-X_5 - \frac{2}{5}X_{12}$ | 0 | 0 | 0 | 0 | $-X_{10}$ | 0 | 0 |
| $X_5$ | 0 | 0 | 0 | $X_5 + \frac{2}{5}X_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $X_{10}$ |
| $X_6$ | $X_2$ | 0 | $X_6$ | 0 | 0 | 0 | $X_8$ | 0 | $X_{11}$ | 0 | 0 | 0 |
| $X_7$ | 0 | $X_{11}$ | 0 | 0 | 0 | $-X_8$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_8$ | $-X_{11}$ | 0 | $X_8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_9$ | 0 | $-X_8$ | 0 | 0 | 0 | $-X_{11}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_{10}$ | 0 | 0 | 0 | $X_{10}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_{11}$ | 0 | 0 | $X_{11}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $X_{12}$ | 0 | 0 | 0 | 0 | $-X_{10}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

## 3.4 Approximately Invariant Solutions

Consider Eq.(3.2),

$$u_{tt} - \epsilon u_t = (uu_x)_x + (uu_y)_y \tag{3.14}$$

Using the symmetry

$$X_4 = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} + \frac{\epsilon}{10}\left(t^2\frac{\partial}{\partial t} - 4ut\frac{\partial}{\partial u}\right) \tag{3.15}$$

The approximate invariants for the operator (3.15) are written in the form

$$E(t,x,y,u,\epsilon) = E^0(t,x,y,u) + \epsilon E^1(t,x,y,u) + o(\epsilon)$$

They are determined by the equation $X(E) = o(\epsilon)$. Using for the operator (3.15) the notation

$$X = X^0 + \epsilon X^1,$$

Such that

$$X^0 = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}, \quad X^1 = \frac{1}{10}\left(t^2\frac{\partial}{\partial t} - 4tu\frac{\partial}{\partial u}\right).$$

We will write the determining equation $X(E) = o(\epsilon)$ for the approximate invariants in the form

$$X^0(E^0) + \epsilon\left[X^0(E^1) + X^1(E^0)\right] = 0, \qquad X^0(E^0) = 0, \quad X^0(E^1) + X^1(E^0) = 0$$

Or

$$\begin{aligned} t\tfrac{\partial E^0}{\partial t} - 2u\tfrac{\partial E^0}{\partial u} &= 0, \\ t\tfrac{\partial E^1}{\partial t} - 2u\tfrac{\partial E^1}{\partial u} &= -\tfrac{1}{10}\left(t^2\tfrac{\partial E^0}{\partial t} - 4tu\tfrac{\partial E^0}{\partial u}\right). \end{aligned} \tag{3.16}$$

Solving Eqs.(3.16), we will have two functionally independent invariants

$$\begin{aligned} E_1 &= {E_1}^0(t,x,y,u) + \epsilon {E_1}^1(t,x,y,u) \\ E_2 &= {E_2}^0(t,x,y,u) + \epsilon {E_2}^1(t,x,y,u) \end{aligned} \tag{3.17}$$

for generator (3.15).

**Remark 4** *Functions* (3.16) *are said to be functionally dependent if* $E_2 = \psi(E_1)$*, equivalently we can say*

$${E_2}^0(t,x,y,u) + \epsilon {E_2}^1(t,x,y,u) = \psi({E_1}^0(t,x,u) + \epsilon {E_1}^1(t,x,y,u)) + o(\epsilon)$$

*a function* $\psi$ *holds identically in* $t,x,y,u$*. If Such a function* $\psi$ *does not exist, the functions* (3.16) *are said to be functionally independent. It is verified that if* ${E_1}^0(t,x,y,u)$ *and* ${E_2}^0(t,x,y,u)$ *are functionally independent, then so are the* *functions* (3.17).

The first equation in (3.16) clearly has two functionally independent solutions, that is,

$$E_1{}^0 = xy, \quad E_2{}^0 = t^2 u.$$

Substituting $E_1{}^0 = xy$ into the second equation in (3.16) and taking its simplest solution $E_1{}^1 = 0$, we found one invariant in (3.17),

$$E_1 = xy. \tag{3.18}$$

Note that it does not involve the dependent variable $u$. Now we substitute the solution $E_2{}^0 = t^2 u$ of the first equation in (3.16) into the second equation in (3.16) and get non-homogeneous linear equation:

$$t\frac{\partial E_1{}^1}{\partial t} - 2u\frac{\partial E_2{}^1}{\partial u} = \frac{1}{5}(t^3 u).$$

The corresponding characteristic equation are

$$\frac{dt}{t} = -\frac{du}{2u} = 5\frac{dE_2{}^1}{t^2 u}$$

The first integral yields $t^2 u = \lambda = const$. Therefore, the second equation

$$\frac{dt}{t} = 5\frac{dE_2{}^1}{t^3 u}$$

gives

$$E_2{}^1 = \frac{1}{5}t^3u + c. \tag{3.19}$$

Assuming that $c = 0$, we obtain the second invariant in (3.17),

$$E_2 = t^2u + \frac{\epsilon}{5}t^3u. \tag{3.20}$$

By remark 4 invariants (3.18), (3.20) are functionally independent. Letting $E_2 = \phi(E_1), i.e.,$

$$\left(t^2u + \frac{\epsilon}{5}t^3u\right) = \phi(xy)$$

and solving for $t^2u$ in the first order of precision,

$$t^2u = \left(1 + \frac{\epsilon}{5}t\right)^{-1}\phi(xy) = \left(1 - \frac{\epsilon}{5}t\right)\phi(xy) + o(\epsilon),$$

thus, the approximately invariant solution is:

$$u(t, x, y) = (\frac{1}{t^2} - \frac{\epsilon}{5t})\phi(xy). \tag{3.21}$$

Differentiation of (3.21) gives:

$$u_t = \left(-\frac{2}{t^3} + \frac{\epsilon}{5t^2}\right)\phi, \qquad u_{tt} = \left(\frac{6}{t^4} - \frac{2\epsilon}{5t^3}\right)\phi,$$

$$u_y{}^2 = \left(\frac{1}{t^4} - \frac{2\epsilon}{5t^3}\right)\left(\frac{\partial\phi}{\partial y}\right)^2, \quad uu_{yy} = \left(\frac{1}{t^4} - \frac{2\epsilon}{5t^3}\right)\frac{\partial^2\phi}{\partial y^2}.\phi,$$

$$u_x{}^2 = \left(\frac{1}{t^4} - \frac{2\epsilon}{5t^3}\right)\left(\frac{\partial\phi}{\partial x}\right)^2, \quad uu_{xx} = \left(\frac{1}{t^4} - \frac{2\epsilon}{5t^3}\right)\frac{\partial^2\phi}{\partial x^2}.\phi,$$

Substituting all the previuos derivatives of u into Eq.(3.2), yields

$$u_{tt}-(uu_x)_x-(uu_y)_y+\epsilon u_t = \left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{\partial^2\phi}{\partial x^2}.\phi + \left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{\partial^2\phi}{\partial y^2}.\phi - 6\phi\right)\left(\frac{1}{t^4} - \frac{2\epsilon}{5t^3}\right) = 0. \tag{3.22}$$

Thus, Eq.(3.22) yields

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{\partial^2\phi}{\partial x^2}.\phi + \left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{\partial^2\phi}{\partial y^2}.\phi - 6\phi = 0. \tag{3.23}$$

Therefore, we need to solve Eq.(3.23) to find the function $\phi(xy)$.

**Case I:** Considering the function $\psi(xy)$ of the form $\phi(xy) = (xy)^\alpha$, Then substituting back into Eq.(3.23), one obtains

$$[2\alpha^2 - \alpha]\left(x^{\alpha-2}y^\alpha + x^\alpha y^{\alpha-2}\right) - 6 = 0$$

For $\alpha = 2$, this implies $\psi(xy) = (xy)^2$ for all $x, y$ such that $x^2 + y^2 = 1$ , thus, the solution in this case has the form

$$u(t,x,y) = (\frac{1}{t^2} - \frac{\epsilon}{5t})(x^2y^2) \quad s.t \quad x^2 + y^2 = 1.$$

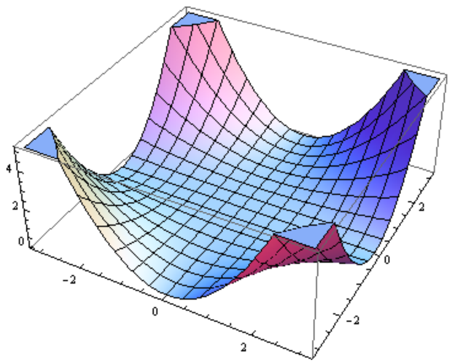


Figure 3.1: Case(i): Approximate invariant solution for $t = \pi, \quad -\pi \leqslant x \leqslant \pi, \quad -\pi \leqslant y \leqslant \pi, \quad \epsilon = 0.1.$

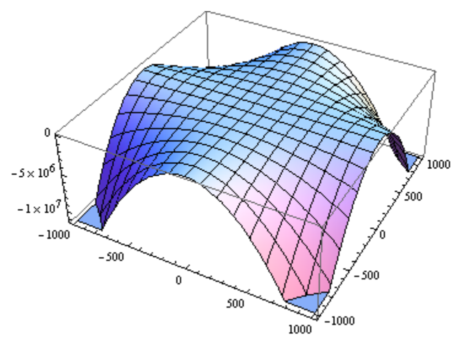


Figure 3.2: Case(ii): Approximate invariant solution for $t = 1000, \quad -1000 \leqslant x \leqslant 1000, \quad 1000 \leqslant y \leqslant 1000, \quad \epsilon = 0.1.$

**Case II:** Considering the function $\phi(xy)$ of the form $\phi(xy) = A(x)B(x)$, Then substituting the derivatives back into Eq.(3.23), one obtains the following equation

$$B\left(\frac{A'^2 + AA''}{A}\right) + A\left(\frac{B'^2 + BB''}{B}\right) - 6 = 0 \tag{3.24}$$

Where

$$A' = \frac{\partial A}{\partial x}, \quad A'' = \frac{\partial^2 A}{\partial x^2}, \quad B' = \frac{\partial B}{\partial y}, \quad B'' = \frac{\partial^2 B}{\partial y^2}$$

Differentiating Eq.(3.24) with respect to $x$, gives

$$B\left(\frac{A^{'2}+AA^{''}}{A}\right)^{'}+A^{'}\left(\frac{B^{'2}+BB^{''}}{B}\right)=0 \tag{3.25}$$

Multiply Eq.(3.25) by $\frac{1}{A'B}$ gives

$$\frac{1}{A'}\left(\frac{A^{'2}AA^{''}}{A}\right)^{'}=-\left(\frac{B^{'2}+BB^{''}}{B}\right) \tag{3.26}$$

Differentaiting both sides in Eq.(3.26) with respect to $x$, implies

$$\left(\frac{1}{A'}\left(\frac{A^{'2}AA^{''}}{A}\right)^{'}\right)^{'}=0 \tag{3.27}$$

which reduces to the following nonlinear ODE

$$AA^{''}+A^{'2}-d_1A^2-c_1A=0, \quad where \;\; d_1,c_1 \;\; are \;\; constants. \tag{3.28}$$

Similarly, from Eq.(3.24) one may also obtain the following nonlinear ODE

$$BB^{''}+B^{'2}-d_2B^2-c_2B=0, \quad where \;\; d_2,c_2 \;\; are \;\; constants. \tag{3.29}$$

Considering Eq.(3.28), let $\gamma(x)=A^2(x)$, implies

$$\gamma^{'}=2AA^{'}, \qquad \gamma^{''}=2AA^{''}+2A^{'2}$$

Substituting these into Eq.(3.28), we obtain

$$\frac{d^2\gamma(x)}{dx^2} - 2d_1\gamma(x) - 2c_1\sqrt{\gamma(x)} = 0.$$

Thus,

$$\frac{d^2\gamma(x)}{2d_1\gamma(x) + 2c_1\sqrt{\gamma(x)}} = d^2x \tag{3.30}$$

Integrating both sides in (3.30) twice and subtituting back $\gamma(x) = A^2(x)$, one obtain the following equation

$$(2\ln(d_1A(x) + c_1) - 1)\left(\frac{A^2(x)}{2d_1} - \frac{{c_1}^2}{2{d_1}^2}\right) + \frac{c_1}{{d_1}^2}A^2(x) = \frac{x^2}{2} + c_3x + c_4, \tag{3.31}$$

where $c_3, c_4$ are arbitrary constants.

Similarly, from Eq.(3.29), one obtains

$$(2\ln(d_2B(x) + c_2) - 1)\left(\frac{B^2(x)}{2d_2} - \frac{{c_2}^2}{2{d_2}^2}\right) + \frac{c_2}{{d_2}^2}B^2(x) = \frac{x^2}{2} + c_5x + c_6, \tag{3.32}$$

where $c_5, c_6$ are arbitrary constants. Therefore, a solution in this case of the form

$$u(t,x,y) = (\frac{1}{t^2} - \frac{\epsilon}{5t})A(x)B(x). \tag{3.33}$$

# CHAPTER 4

# NONLINEAR (2+1) DIMENSIONAL WAVE EQUATION WITH A FORCING TERM

## 4.1 Introduction

In this chapter we study ways of finding the approximate symmetries of nonlinear (2+1) dimensional wave equation with a forcing term using two of the approximate Lie symmetry methods. We compare these different methods and discuss advantages of using one over the other. Moreover approximate invariant solutions of the nonlinear wave equation with a forcing term based on the Lie group method are constructed.

The nonlinear (2+1) dimensional wave equation with a forcing term we consider, is given by:

$$u_{tt} - (uu_x)_x - (uu_y)_y = \epsilon f(u) \tag{4.1}$$

## 4.2 Finding the Approximate Symmetries using METHOD I

This method was described in chapter 1 and subsequently used in chapter 2. We first need to find the exact symmetries of the unperturbed part of Eq.(4.1),

$$u_{tt} - (uu_x)_x - (uu_y)_y = 0. \tag{4.2}$$

These are given as (see section 3.2)

$$\begin{aligned} \xi_0 &= a_3x + a_1y + a_2 \\ \theta_0 &= a_3y - a_1x + a_6 \\ \tau_0 &= a_4t + a_5 \\ \eta_0 &= 2u(a_3 - a_4) \end{aligned} \tag{4.3}$$

where $a_1, a_2, a_3, a_4, a_5$and $a_6$ are arbitrary constants. The second step, is to find the auxilary function

$$H = \frac{1}{\epsilon}\left[X^0(F_0(z) + \epsilon F_1(z)\right] |_{F_0(z)+\epsilon F_1(z)=0}, \tag{4.4}$$

where

$$X^0 = \tau_0 \frac{\partial}{\partial t} + \xi_0 \frac{\partial}{\partial x} + \theta_0 \frac{\partial}{\partial y} + \eta_0 \frac{\partial}{\partial u}$$

Substituting the expression (4.3) of the generator $X_0$ into above equation (4.1), one may obtain the auxiliary function

$$H = -2a_3 \left( uf'(u) + f(u) \right) + 2a_4 \left( uf'(u) - 2f(u) \right). \tag{4.5}$$

The third Step is to calculate the operator $X^1$ by requiring that

$$X^1(F_0(z))|_{F_0(z)=0} + H = 0. \tag{4.6}$$

This condition can be written as

$$\left[ {X_1}^{(2)}(u_{tt} - (uu_x)_x - (uu_y)_y)|_{u_{tt}=(uu_x)_x+(uu_y)_y} \right] - 2c_3 \left( uf'(u) + f(u) \right) + 2c_4 \left( uf'(u) - 2f(u) \right) = 0 \tag{4.7}$$

where ${X_1}^{(2)}$ is the second prolongation of $X_1$.

The left hand side of Eq.(4.7) becomes a polynomial in the variables $u_t, u_x, u_y, u_{xx}, u_{yy}, u_{tt}, u_{xy}, u_{xt}, u_{ty}$. Equating to zero the coefficients of these quantities, we obtain,

e1: ${\xi_1}_u = 0$

e2: ${\xi_1}_t = 0$

e3: ${\tau_1}_u = 0$

e4: $\tau_{1y} = 0$

e5: $\tau_{1x} = 0$

e6: $\theta_{1u} = 0$

e7: $\theta_{1t} = 0$

e8: $\eta_{1uu} = 0$

e9: $-2\eta_{1x} - 2u\eta_{1xu} + u\xi_{1xx} + u\xi_{1yy} = 0$

e10: $-\eta_{1u} + 2\xi_{1x} - 2\tau_{1t} = 0$

e11: $-2\eta_{1y} + u\theta_{1xx} + u\theta_{1yy} = 0$

e12: $-2\eta_{1y} + 2\theta_{1y} + \eta_{1u} - 2\tau_{1t} = 0$

e13: $-u\eta_{1xx} + \eta_{1tt} - u\eta_{1yy} - 2a_3(uf^{'}(u) + f(u)) = 0$

e14: $2\eta_{1ut} - \tau_{1tt} = 0$

e15: $2u\xi_{1x} - 2u\tau_{1t} - \eta_1 = 0$

e16: $\theta_{1x} + \xi_{1y} = 0$

e17: $-2u\tau_{1t} + 2u\theta_{1y} - \eta_1 = 0$

Form (e1) and (e2), implies

e18: $\xi_1 = G(x, y)$

From (e3),(e4) and (e5),obtains

e19: $\tau_1 = K(t)$

Using (e5) and (e6),gives

e21: $\theta_1 = L(x, y)$

Differentiate (e12) with respect to $u$, gives $-\eta_{0uy} = 0$, which is implies

e21: $\eta_1 = \varphi_1(t, x)u + \varphi_2(t, x)$

Differentiate (e17) with respect to $u, y$ respectively, implies $\theta_{1yy} = 0$

Differentiate (e10) with respect to $y$, gives $\xi_{1xy} = 0$

From (e11),after differentiating it with respect to $x$, deduces $\theta_{1xx} = 0$.

From (e15) and (e10), subtracting one from the other ,gives

e22: $\xi_{1x} - \theta_{1y} = 0$

Differentiate (e16) with respect to $y$ and Differentiate (e22) with respect to $x$, then subtract the resulting equations one from the other, gives

e23: $\xi_{1xx} + \xi_{1yy} = 0$

Using (e9) and (e23), obtains $\eta_{1xu} = 0$

Differentiate (e15) with respect to $u, x$ respectively, implies $\xi_{1xx} = 0$

Differentiate (e17) with respect to $u, x$ respectively ,obtains

e24: $\theta_{1xy} = 0$

From (e16) and (e24), obtains $\xi_{1yy} = 0$, differentiate (e17) with respect to $u, t$ respectively,implies $\tau_{1tt} = 2\eta_{1ut}$ , substitute it into (e14),one may obtains $\eta_{1tu} = 0$ and $\tau_{1tt} = 0$

Finally, from all the previous,we obtain

$$
\begin{aligned}
\xi_1 &= b_3x + b_1y + b_2 \\
\theta_1 &= b_3y - b_1x + b_6 \\
\tau_1 &= b_4t + b_5 \\
\eta_1 &= 2u(b_3 - b_4)
\end{aligned} \tag{4.8}
$$

where $b_1, b_2, b_3, b_4, b_5$and $b_6$ are arbitrary constants. Then for consistency check , using (4.8) in (e13), we get

$$
-2a_3(uf^{'}(u) + f(u)) = 0 \tag{4.9}
$$

From Eq.(4.9) it can be easily noticed that the following two cases arise, namely,

Case I: $a_3 = 0$

Case II: $uf^{'}(u) + f(u) = 0$

We consider these possibilities one by one.

**Case I:** Reduce that the scaling operator

$$
X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial u} + 2u\frac{\partial}{\partial u}
$$

is not stable, hence, the perturbed (2+1) dimensional nonlinear wave equation (4.1) does not inherit the symmetries of the unperturbed part of the equation.

**Case II:** Solving the first order linear differential equation $uf^{'}(u) + f(u) = 0$, we obtain $f(u) = \frac{k_1}{u}$ , where $k_1$ is constant, then the approximate symmetry generator is given as follow

$$
\begin{aligned}
X = X_0 + \epsilon X_1 = & \left[(a_3+\epsilon b_3)x + (a_1+\epsilon b_1)y + (a_2+\epsilon b_2)\right]\frac{\partial}{\partial x} \\
& + \left[(a_3+\epsilon b_3)y - (a_1+\epsilon b_1)x + (a_6+\epsilon b_6)\right]\frac{\partial}{\partial y} \\
& + \left[(a_4+\epsilon b_4)t + (a_5+\epsilon b_5)\right]\frac{\partial}{\partial t} \\
& + \left[2u((a_3+\epsilon b_3) - (a_4+\epsilon b_4))\right]\frac{\partial}{\partial u}
\end{aligned} \tag{4.10}
$$

These additional symmetries defined in (4.10) are actually the same as those obtained for the unperturbed equation and they are considered as trivial symmetries.

To summarize, in this case METHOD I yields trivial symmetries only.

## 4.3 Finding the Approximate Symmetries using alternative Method

In this, we have expand the dependent variable in a perturbation series. The obtained equation are assumed to be coupled and the approximate symmetry of the original equation is defined as the exact symmetry of these outcoming coupled equations.

Expanding the dependent variable to the first order of $\epsilon$

$$u = v + \epsilon w + o(\epsilon), \quad \epsilon \to 0$$

then, one can expand $f(u) = \frac{k_1}{u}$ in a series in $\epsilon$

$$f(u) = f(v + \epsilon w) = \frac{k_1}{v + \epsilon w} = \frac{k_1}{v}\left[\frac{1}{1 - \left[-\frac{\epsilon w}{v}\right]}\right]$$

Using Taylor expansion in the first order of precision, gives

$$f(u) = \frac{k_1}{v}\left[1 - \frac{\epsilon w}{v} + o(\epsilon)\right] = \frac{k_1}{v} - \frac{\epsilon k_1 w}{v^2} + o(\epsilon), \qquad \epsilon \to 0$$

Substituting the above expansion into Eq.(4.1) and separating at each order of perturbation parameter, one may obtain the following

$$\begin{aligned} v_{tt} - {v_x}^2 - v v_{xx} - {v_y}^2 - v v_{yy} &= 0, \\ w_{tt} - 2v_x w_x - v w_{xx} - w v_{xx} - 2v_y w_y - v w_{yy} - w v_{yy} &= \tfrac{k_1}{v}. \end{aligned} \tag{4.11}$$

Now, the infinitesimal generator for the problem is

$$\begin{aligned} X = \tau(t, x, y, , v, w)\tfrac{\partial}{\partial t} + \xi(t, x, y, v, w)\tfrac{\partial}{\partial x} + \theta(t, x, y, v, w)\tfrac{\partial}{\partial y} \\ +\phi(t, x, y, v, w)\tfrac{\partial}{\partial v} + \eta(t, x, y, v, w)\tfrac{\partial}{\partial w}. \end{aligned} \tag{4.12}$$

Using standard Lie group analysis, the infinitesimals are calculated as follows

$$\begin{aligned}
t &= c_4 t + c_5 \\
\xi &= c_1 x - c_3 y + c_6 \\
\theta &= c_3 x + c_1 y + c_2 \\
\phi &= 2v(-c_4 + c_1) \\
\eta &= -2w(c_1 - 2c_4)
\end{aligned} \tag{4.13}$$

where $c_1, c_2, c_3, c_4, c_5$ and $c_6$ are arbitrary constants. Hence we obtain the following symmetries

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2v\frac{\partial}{\partial v} - 2w\frac{\partial}{\partial w}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$
$$X_4 = t\frac{\partial}{\partial t} - 2v\frac{\partial}{\partial v} + 4w\frac{\partial}{\partial w}, \qquad X_5 = \frac{\partial}{\partial t}, \qquad X_6 = \frac{\partial}{\partial x} \tag{4.14}$$

The following table of commutators, show that (4.14) span a sixth-dimensional Lie algebra.

Table 4.1: Table of Commutators Span Sixth dimensional Lie algebra

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ |
|---|---|---|---|---|---|---|
| $X_1$ | 0 | $-X_2$ | 0 | 0 | 0 | $-X_6$ |
| $X_2$ | $X_2$ | 0 | $-X_6$ | 0 | 0 | 0 |
| $X_3$ | 0 | $X_6$ | 0 | 0 | 0 | $-X_2$ |
| $X_4$ | 0 | 0 | 0 | 0 | $-X_5$ | 0 |
| $X_5$ | 0 | 0 | 0 | $X_5$ | 0 | 0 |
| $X_6$ | $X_6$ | 0 | $X_2$ | 0 | 0 | 0 |

## 4.4 Approximate Invariant Solution

Using the symmetry $X_3$ from (4.14)

$$X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

the characteristic equations are given by

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dv}{0} = \frac{dw}{0} \tag{4.15}$$

Considering $\frac{dx}{-y} = \frac{dy}{x}$ from (4.15) and integrating gives $\alpha = x^2+y^2$. The remaining part of characteristic equation (4.15) suggests that $w = w(\alpha), \quad v = v(\alpha)$. Now we re-write the coupled equations (4.11) in terms of the new variable $\alpha$. This can be done by expressing derivatives of the dependent variables $v$ and $w$ in term of each $x$ and $y$ given by

$$v_t = v_\alpha\frac{\partial\alpha}{\partial t} = 0, \qquad v_{tt} = 0$$

$$v_x = v_\alpha\frac{\partial\alpha}{\partial x} = 2xv_\alpha, \qquad v_{xx} = 2v_\alpha + 2xv_{\alpha\alpha}\frac{\partial\alpha}{\partial x} = 2v_\alpha + 4x^2v_{\alpha\alpha}$$

$$v_y = v_\alpha\frac{\partial\alpha}{\partial y} = 2yv_\alpha, \qquad v_{yy} = 2v_\alpha + 2yv_{\alpha\alpha}\frac{\partial\alpha}{\partial y} = 2v_\alpha + 4y^2v_{\alpha\alpha}$$

$$w_t = w_\alpha\frac{\partial\alpha}{\partial t} = 0, \qquad w_{tt} = 0$$

$$w_x = w_\alpha\frac{\partial\alpha}{\partial x} = 2xw_\alpha, \qquad w_{xx} = 2w_\alpha + 2xw_{\alpha\alpha}\frac{\partial\alpha}{\partial x} = 2w_\alpha + 4x^2w_{\alpha\alpha}$$

$$w_y = w_\alpha\frac{\partial\alpha}{\partial y} = 2yw_\alpha, \qquad w_{yy} = 2w_\alpha + 2yw_{\alpha\alpha}\frac{\partial\alpha}{\partial y} = 2w_\alpha + 4y^2w_{\alpha\alpha}$$

Using this into equation (4.11) we obtain second order ordinary differential equations

$$\begin{aligned} \alpha {v_\alpha}^2 + \alpha v v_{\alpha\alpha} + v v_\alpha &= 0 \\ 2\alpha v_\alpha w_\alpha + v w_\alpha + \alpha v w_{\alpha\alpha} + w v_\alpha + \alpha w v_{\alpha\alpha} &= \tfrac{k_1}{4v} \end{aligned} \tag{4.16}$$

Using the substitution $v(\alpha) = \frac{H(\alpha)^2}{2}$ into the first eqution in (4.16) , one may obtains

$$\frac{\partial H(\alpha)}{\partial \alpha} + \alpha \frac{\partial^2 H(\alpha)}{\partial \alpha^2} = 0$$

whose solution is $H(\alpha) = c_1 \ln \alpha + c_2$ where $c_1$ and $c_2$ are arbitrary constants of integration. Rewriting this solution as function of $v$, we get $v(\alpha) = \sqrt{c_2 + c_1 \ln \alpha}$. Then substituting the derivatives of $v(\alpha)$ into the second equation in (4.16) and considering $c_2 = 0, c_1 = 1$, reduces to second-order ordinary differential equation

$$(1 + \ln \alpha)\, w_\alpha + (\alpha \ln \alpha) w_{\alpha\alpha} - \frac{1}{4 \ln \alpha} w = \frac{k_1}{4} \tag{4.17}$$

We know such a solution may be hard to obtain. However, we may obtain an asymptotic estimate of the solution of Eq.(4.17) using the symptotic expansions [18].

**Definition 2** *We say that the function $f(x) = O\left(g(x)\right)$ as $x \to x_0$ if $f$ and $g$ are such that $\lim_{x \to x_0} f/g = C$, $C$ constant.*

Thus, in Eq.(4.17), we find $(1 + \ln \alpha) = O(\alpha)$, $(\alpha \ln \alpha^2) = O(\alpha^2)$ and $\frac{1}{4 \ln \alpha} = O(1)$ as $\alpha \to \infty$. Therefore Eq. (4.17) is asymptotically equivalent to the following equation for large $\alpha$

$$\alpha^2 w_{\alpha\alpha} + \alpha w_\alpha + w = \frac{k_1}{4} \tag{4.18}$$

Then one may easily obtains the solution of the above non-homogeneous Cauchy-Euler equation as

$$w(\alpha) = k_2 \sin(\ln \alpha) + k_3 \cos(\ln \alpha) + \frac{k_1}{4}$$

where $k_2, k_3$ are constants. Lastly, we re-cast the solution in original coordinates as

$$\begin{aligned} u(t,x,y) &= v(t,x,y) + \epsilon w(t,x,y) \\ &= \sqrt{\ln(x^2+y^2)} + \epsilon \left( k_2 \sin(\ln(x^2+y^2)) + k_3 \cos(\ln(x^2+y^2)) + \tfrac{k_1}{4} \right) \end{aligned} \tag{4.19}$$

This is an approximate solution invariant under rotation in x-y, dilation in space and u coordinates.

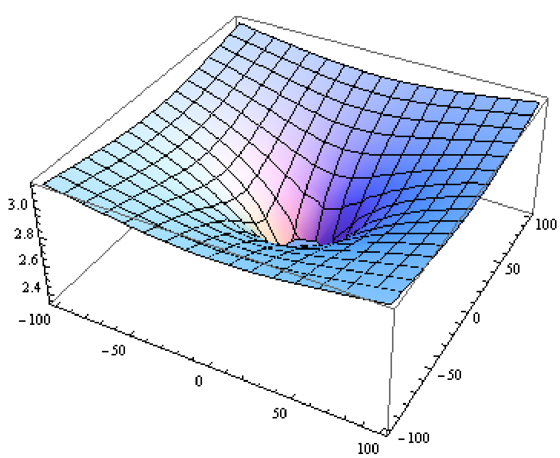


Figure 4.1: (a) Invariant Solution of the Unperturbed Equation (4.2) for $-100 \leqslant x \leqslant 100, \quad -100 \leqslant y \leqslant 100, \quad x^2 + y^2 \geqslant 1$.

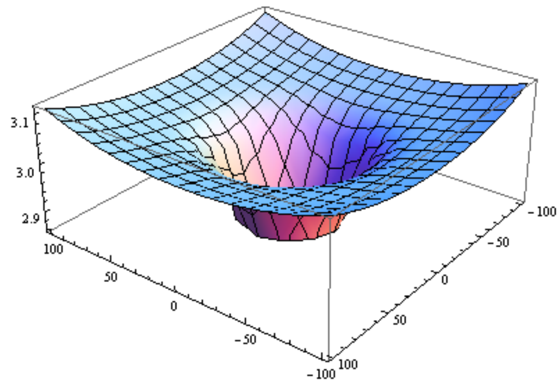


Figure 4.2: (b) Approximate Invariant Solution of the Equation (4.1) for $-100 \leqslant x \leqslant 100, \ \ -100 \leqslant y \leqslant 100, \ \ and \ \ x^2+y^2 \geqslant 1, \ \ k_1 = 4, \ \ k_2 = 1, \ \ k_3 = 1, \ \ \epsilon = 0.1.$

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